Chapter 4. Fuzzy Logic and Approximate Reasoning

- ***Linguistic variable:***

  ~ A variable whose values are words or sentences in a natural or artificial language.

  Ex: "speed", values: slow, fast, very fast.

  ~ providing a means of approximate characterization of phenomena that are too complex or too ill-defined to be amenable to description in conventional quantitative terms.

- ***Fuzzy variable:***

  A fuzzy variable is characterized by a triple(X, U, R(x))

  X：the name of the variable.

  U：universe of discourse.

  R(x)：a fuzzy subset of U which represents a fuzzy restriction imposed by X.

Ex: X= “old”.

U={10, 20, …, 80}

R(x)=0.1/20+0.2/30+0.4/40+0.5/50+0.8/60+1/70+1/80

- A linguistic variable is a variable of higher order than a fuzzy variable, and it takes fuzzy variable as its values. A linguistic variable is characterized by a quintuple (x, T(x), U, G, M).

  x：name of the variable.

  T(x)：a term set of x, i.e. the set of names of linguistic values of x with each value being a fuzzy variable defined on U.

  G：a syntactic rule for generating the name of values of x.

  M： a semantic rule for associating each value of x with its meaning.

Ex 6.1:

x="speed"

T(speed)={very slow, slow, Moderate, Fast, …}

G: intuitive.

M(slow): the fuzzy set for "a speed below about 40 miles per hour (mph)" with μ_{slow}

M(Moderate):the fuzzy set for "a speed close to 55mph" with $\mu_{Moderate}$

- A linguistic variable x is said to be *structured* if its term set T(x) and its semantic rule M can be characterized such that they can be viewed as algorithmic procedures for generating the elements of T(x) and computing the meaning of each term in T(x).
- ***Linguistic hedge (or modifier) h:***

An operator for modifying the meaning of its operator or, more generally, of a fuzzy set A to create a new fuzzy set h(A).

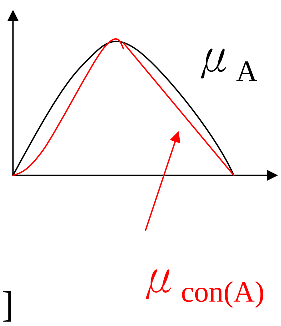
- The following fuzzy set operations are used in defining a linguistic hedge:

~ Concentration：con(A)

$$\mu_{CON(A)}(u) = (\mu_A(u))^2$$

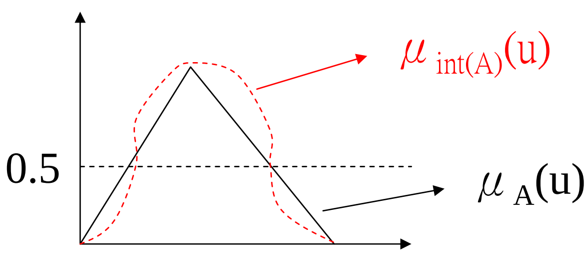
~ Dilation：dil(A)

$$\mu_{dil(A)}(u) = (\mu_A(u))^{\frac{1}{2}}$$



~ Intensification：int(A).

$$\mu_{int(A)}(u) = \begin{cases} 2(\mu_A(u))^2, & \mu_A(u) \in [0,0.5] \\ 1-2(1-\mu_A(u))^2, & \text{otherwise} \end{cases}$$



- some popular linguistic hedges are：

$$\text{Very}(A) = \text{con}(A): (\mu_A(u))^2$$
$$\text{highly}(A): (\mu_A(u))^3$$
$$\text{fairly(more or less)}(A) = \text{dil}(A): (\mu_A(u))^{\frac{1}{2}}$$
$$\text{roughly}(A)=\text{dil}(\text{dil}(A))$$
$$\text{plus}(A): (\mu_A(u))^{1.25}$$
$$\text{minus}(A) = (\mu_A(u))^{0.75}$$
$$\text{rather}(A)=\text{int}(\text{con}(A)) \text{ AND NOT}[\text{con}(A)].$$

where AND and NOT are the fuzzy conjunction and complement.

- The resulting fuzzy sets should be normalized if the height is not one.
- With the aid of linguistic hedges, we can define a term set of a linguistic variable, such as

T(Age)={old,very old,very very old, ...}

in which "old" is called the primary term, and the corresponding syntactic rule G which can generate the term set, T(Age), recursively, could be the following recursive algorithm：

$$T^{i+1}=\{old\}\cup\{very\ T^i\}.\ i=0,1,2,........(*)$$

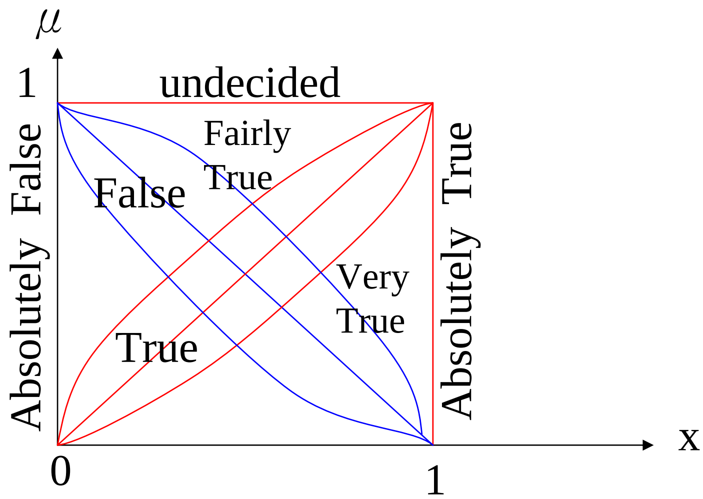
Ex: for i = 0, 1, 2, 3. we have

$$T^0 = \varnothing$$
$$T^1 = \{old\}.$$
$$T^2 = \{old\ ,\ very\ old\}.$$
$$T^3 = \{old\ ,\ very\ old\ ,\ very\ very\ old\}.$$

- The semantic rule M which can associate with each T^i a meaning could be the following recursive algorithm：

$$M(T^i) = \bigcup old^j,\ j = 1, 2, \ldots, 2^{i-1}. \qquad (**)$$

with Eqs. (*) and (**), "Age" is a structured linguistic variable.



• ***Linguistic Approximation***

~ A procedure for solving the problem of how to find a term from the term set of a linguistic variable such that the meaning of this term is closest to a given fuzzy set.

~ To solve the linguistic approximation problem, one can base on the *similarity measure* of two fuzzy sets, E(A, B), which indicates the degree of equality of two fuzzy sets A and B,

$$E(A, B) = \frac{|A \cap B|}{|A \cup B|}$$

where $0 \le E(A,B) \le 1$, and $|A| = \sum \mu_A(x)$.

~ The idea is that the linguistic approximation of a given fuzzy set A is the term $T_A \in T(x)$, which is most similar to A than the other terms in the term set T(x), of a given linguistic variable, that is

$$E(A, T_A) = \max_{T^i \in T(x)} E(A, T^i)$$

Ex 6.2：

Given U={0, 0.1, 0.2, ..., 1}, and the term set {A, B, C} of the linguistic variable "Truth", where

A = "True" = 0.7/0.8+1/0.9+1/1

B = "MORE OR LESS True" = 0.5/0.6+0.7/0.7+1/0.8+1/0.9+1/1

C = "ALMOST True" = 0.6/0.8+1/0.9+0.6/1

find the linear approximation of the fuzzy set D defined by

D=0.6/0.8+1/0.9+1/1

since

$$E(D,A)=\frac{|D\cap A|}{|D\cup A|}=\frac{0.6+1+1}{0.7+1+1}=\frac{2.6}{2.7}=0.96$$

$$E(D,B)=\frac{|D\cap B|}{|D\cup B|}=\frac{0.6+1+1}{0.5+0.7+1+1+1}=\frac{2.6}{4.2}=0.62$$

$$E(D,C)=\frac{|D\cap C|}{|D\cup C|}=\frac{0.6+1+0.6}{0.6+1+1}=\frac{2.2}{2.6}=0.85$$

the linguistic approximation of the fuzzy set D is the term A "True".

- ***Fuzzy Logic*** **:**
- Logic is a basis for reasoning.
- Classical bivalence logic deals with propositions that are required to be either true (with a logical value of 1) or false (with a logical value of 0) , which are called truth value of the propositions.
- Propositions are sentences expressed in some language and can be expressed, in general, in the canonical form

x is P,

where x is a symbol of a subject, and P designates a predicate which characterizes a property of the subject.

- For example, "Taipei is in Taiwan", is a proposition in which "Taipei" is a subject and "in Taiwan" is a predicate that specifies a property of "Taipei" , namely, its geographical position in Taiwan.
- Each proposition A has an opposite called *negation* of the proposition and is denoted as $\overline{A}$

- A proposition and its negation are required to assume opposite truth values.
- We shall consider two major topics in logic:

  logic operations and *logic reasoning*.

- *Logic operations*

~ Logic operations are (logic) functions of two propositions.

~ The logic operations are defined via truth tables.

~ Consider two propositions A and B, either of which can be true ("1") or false ("0").

~ The four basic logic operations are "*conjunction*($\wedge$)", "*disjunction*($\vee$)", "*implication* (or *conditional*)(=>)", and "*equivalence* (or *bi-directional*)(<=>)" and are interpreted as "A and B", "A or B", "if A then B", and "A if and only if B," respectively.

- Truth table

Propositions		Conjunction	Disjunction	Implication	Equivalence
A	B	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
1	1	1	1	1	1
1	0	0	1	0	0
0	1	0	1	1	0
0	0	0	0	1	1

- ***Logic reasoning***

~ Another main concern of logic system is the reasoning procedure which is performed through some *inference rules.*

~ Some important inference rules are：

$(A \wedge (A \Rightarrow B)) \Rightarrow B$ (modus ponens)

(*) $(\overline{B} \wedge (A \Rightarrow B)) \Rightarrow \overline{A}$ (modus tollens)

$((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$ (hypothetical syllogism)

~ The resulting propositions in Eq(*) are always true no matter what the truth values of propositions A and B are. These are called tautologies.

~ We interpret the modus ponens as: If A is true and if the proposition “If A is true then B is true” is also true then proposition B is true.

~ The modus ponens is closely relaed to the forward data-driven inference which, when extend to fuzzy logic, is particularly useful for fuzzy logic control.

- Unlike two-valued logic, the truth values of propositions in fuzzy logic are allowed to range over the fuzzy subsets of unit interval [0, 1], or more simply, a point in the interval.
- For example, a truth value in fuzzy logic "very true" may be interpreted as a fuzzy set in [0, 1].
- Def：The *truth-value* of the proposition "x is A," or simply truth value of A, which is denoted by ν(A) is defined to be a point in [0, 1] (called *numerical truth-value*) or possibly a fuzzy set in [0, 1] (called *linguistic truth- value*).

- Let us see how to get the truth value of a proposition which comes from the logic operations (that is , negation, conjunction, disjunction, and implication) of other propositions whose truth values are known.
- Def：

If ν(A) and ν(B) are numerical truth values of propositions A and B, respectively, then

$$\nu(\text{NOT } A) \triangleq 1-\nu(A)$$
$$\nu(A \text{ AND } B) \triangleq \nu(A)\wedge\nu(B) = \min\{\nu(A),\nu(B)\}$$
$$\nu(A \text{ OR } B) \triangleq \nu(A)\vee\nu(B) = \max\{\nu(A),\nu(B)\}$$
$$\nu(A \Rightarrow B) \triangleq \nu(A)\Rightarrow\nu(B) = \max\{1-\nu(A),\min(\nu(A),\nu(B))\}.$$

(other definitions of implications can also be applied (See Table 7.1)).

- Using the extension principle, we have the following definition.

- Def：If ν(A) and ν(B) are linguistic truth-values of propositions A and B , respectively, and are expressed as

$$\nu(A)=\alpha_1/\nu_1+\alpha_2/\nu_2+..........+\alpha_n/\nu_n.$$
$$\nu(B)=\beta_1/w_1+\beta_2/w_2+..........+\beta_m/w_m.$$

where α_i , β_i , ν_i , w_i $\in$ [0 , 1], then we have

$$\nu(\text{NOT A}) = \alpha_1/(1-\nu_1)+\alpha_2/(1-\nu_2) + ... + \alpha_n/(1-\nu_n)$$

$$\nu(\text{A AND B}) = \nu(A)\wedge\nu(B) = \sum_{i,j}\min(\alpha_i , \beta_j)/\min(\nu_i , w_j).$$

$$\nu(\text{A OR B}) = \nu(A)\vee\nu(B) = \sum_{i,j}\min(\alpha_i , \beta_j)/\max(\nu_i , w_j).$$

$$\nu(A\Rightarrow B) = \nu(A)\Rightarrow\nu(B)$$
$$= \sum_{i,j}\min(\alpha_i , \beta_j)/\max\{1 - \nu_i , \min(\nu_i ,w_j)\}....(*)$$

Remark：Various definitions of implication can be applied to Eq (*).

Ex6.3：ν(P)="MORE OR LESS True", ν(Q)="ALMOST True", are linguistic values of propositions P and Q, and are defined in Ex 6.2.

ν(NOT P) = ν(NOT MORE OR LESS True)

= 0.5/0.4+0.7/0.3+1/0.2+1/0.1+1/0

ν(P AND Q) = ν(P) Λ ν(Q)

= 0.5/0.6+0.7/0.7+1/0.8+1/0.9+0.6/1

"MORE OR LESS True" = 0.5/0.6+0.7/0.7+1/0.8+1/0.9+1/1
"ALMOST True" = 0.6/0.8+1/0.9+0.6/1

• ***Approximate Reasoning***

~ As in any other logic, the rule of inference in fuzzy logic governs the deduction of a proposition, q, from set of premises $\{p_1, p_2, ..., p_n\}$

~ In fuzzy logic both the premises and the conclusion are allowed to be a fuzzy proposition.

~ Since the inferenced results usually must be translated into more meaningful terms (fuzzy sets) by the use of linguistic approximation, the final conclusion drawn from the premises $p_1, p_2, ..., p_n$ is, in general, an approximate rather than exact consequences of $p_1, p_2, ..., p_n$.

~ There are four principal modes of fuzzy reasoning (or approximate reasoning) in fuzzy logic. *categorical reasoning, qualitative reasoning, syllogistic reasoning* and *dispositional reasoning*.

~ We shall only discuss categorical reasoning and qualitative reasoning which are used in FLCs.

· *Categorical reasoning*:

~ In this mode of reasoning, the premises are assumed to be in the canonical form " X is A " or in the conditional canonical form , "If X is A then Y is B", where A and B are fuzzy predicates.

~ Notations：

X, Y, Z, ... = (fuzzy) variables taking values in the universes U,V, W.

A, B, C, ... = fuzzy predicates.

~ Projection rule of inference:

$$\frac{(X,Y) \text{ is } R}{X \text{ is } [R\downarrow X]}$$

where $[R \downarrow X]$ denotes the projection of fuzzy relation R on X.

Ex:

$$\frac{(X, Y) \text{ is Close to } (3, 2)}{X \text{ is Close to } 3.}$$

~ Conjunction or particularization rule of inference:

$$\frac{\begin{array}{c} X \text{ is } A \\ X \text{ is } B \end{array}}{X \text{ is } A \cap B} \qquad \frac{\begin{array}{c} (X, Y) \text{ is } A \\ X \text{ is } B \end{array}}{(X, Y) \text{ is } A \cap (B \times V)}$$

$$\frac{\begin{array}{c} (X, Y) \text{ is } A \\ (Y, Z) \text{ is } B \end{array}}{(X,Y,Z) = (A \times W) \cap (U \times B)}$$

Ex：

$$\frac{\begin{array}{c} \text{pressure is NOT VERY High} \\ \text{pressure is NOT VERY Low} \end{array}}{\text{pressure is NOT VERY High AND NOT VERY Low}}$$

$$\frac{\begin{array}{c} \text{X and Y are APPROXIMATELY Equal} \\ \text{X is small} \end{array}}{\text{X and Y are (APPROXIMATERLY Equal} \cap \text{(small} \times \text{V)).}}$$

- Disjunction or Cartesian product rule of inference:

$$\frac{\begin{array}{c}X \text{ is } A \\ \text{OR } X \text{ is } B\end{array}}{X \text{ is } A \cup B} \qquad \frac{\begin{array}{c}X \text{ is } A \\ Y \text{ is } B\end{array}}{(X,Y) \text{ is } A \times B}$$

- Negation rule:

$$\frac{\text{NOT}(X \text{ is } A)}{X \text{ is } \overline{A}} \qquad \frac{\text{NOT(John is Tall)}}{\text{John is NOT Tall}}$$

- Entailment rule of inference (entailment principle):

$$\frac{\begin{array}{c}X \text{ is } A \\ A \subset B\end{array}}{X \text{ is } B}$$

Ex：

$$\frac{\begin{array}{c}\text{Mary is VERY Young} \\ \text{VERY Young} \subset \text{Young}\end{array}}{\text{Mary is Young}}$$

- Compositional rule of inference:

$$\frac{\begin{array}{c}X \text{ is } A \\ (X,Y) \text{ is } R\end{array}}{Y \text{ is } A \circ R}$$

$$\mu_{A \circ R}(v) = \max_u \min(\mu_A(u), \mu_R(u,v)).$$

The compositional rule of inference may be viewed as combination of the conjunction and projection rules.

By conjunction rule: (X, Y) is R ∩ (A × V)

By projection rule: Y is [(R ∩ (A × V)) ↓ Y]

· Generalized modes ponens:

$$\frac{\begin{array}{c}\text{X is A}\\ \text{If X is B, THEN Y is C}\end{array}}{\text{Y is A o } (\bar{B} \oplus C)}$$

Material implication(1-38):

$$B \rightarrow C = s(\bar{B}, C) = (\bar{B} \oplus C)$$

s: bounded sum

where $\mu_{\bar{B} \oplus C}(u, v) = \min(1, 1 - \mu_B(u) + \mu_c(v)).$

~ Actually, the generalized modus ponens is a special case of the compositional rule of inference.

~ Unlike the modus ponens in classical logic, the generalized modus ponens does not require that the precondition "X is B" be identical to the premise "X is A".

~ The generalized modus ponens is related to the interpolation rule for solving the important problem of "partial" matching of preconditions of rules that arises in the operation of any rule-based system.

~ The generalized modus poems plays a very important role in fuzzy control and fuzzy expent systems.

· Extension principle:

$$\frac{\text{X is A}}{\text{f (X) is f (A)}}$$

where f is a mapping from U to V so that X is mapped into f(X). By extension principle,

$$\mu_{f(A)}(v) = \sup_{v = f(u)} \mu_A(u), \quad u \in U, v \in V.$$

The extension principle can answer a question like, assuming that X taking values in U is constrained by the proposition “X is A,” what is the constraint on f(x) that is induced by the constraint on X ? For example.

$$\frac{\text{X is Small}}{X^2 \text{ is small}^2\text{, where } \mu_{small^2}(v) = \sup_{v=u^2} \mu_{small}(u).}$$

Ex：Consider the premises

p_1=John is VERY Big.

p_2=John is VERY Tall.

where "Big" is a given fuzzy subset of U×V (i.e. values of Height × values of Weight) and "Tall" is a given fuzzy subset of U (values of Height).

How heavy is John?

Sol: Method1: conjunction rule of inference,

$$\frac{\begin{array}{c}\text{(height (John),weight(John))is Big}^2\\ \text{(height(John)) is Tall}^2\end{array}}{\text{(height(John), weight(John)) is Big}^2 \cap (\text{Tall}^2 \times \text{V})}$$

projection rule of inference.

$$\text{weigh (John) is } [(\text{Big}^2 \cap (\text{Tall}^2 \times V)) \downarrow \text{weight}].$$

Method 2: compositional rule of inference

$$\text{weigh (John) is } \text{Tall}^2 \circ \text{Big}^2$$

Ex：Consider the premises

p_1 = X is small.

p_2 = X and Y are APPROXIMATELY Equal.

in which X and Y range over the set U = {1,2,3,4} and

Small = 1/1+0.6/2+0.2/3

$$\text{APPROXIMATELY Equal:}\quad \begin{array}{cc} & Y \\ X & \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0 \\ 0 & 0.5 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix} \end{array}$$

Y is ?

Sol:

Method 1:

Y = X is small o APPROXIMATELY_ Equal.

= (1, 0.6, 0.5, 0.2).

Y = 1/1 + 0.6/2 + 0.5/3 + 0.2/4

Through the linguistic approximation, we can say, for example, "Y is MORE OR LESS Small," which is inferred from propositions p_1 and p_2.

Method 2: By conjunction and projection rules.

$$Y=[((\text{Small}\times U)\cap \text{APPROXIMATELY Equal})\downarrow Y]$$

$$\text{Small}\times U=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.6 & 0.6 & 0.6 & 0.6 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.6 & 0.6 & 0.6 & 0.6 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cap \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 1 & 0.5 & 0 \\ 0 & 0.5 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 0.6 & 0.5 & 0 \\ 0 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0.5 & 0.6 & 0.5 & 0 \\ 0 & 0.2 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \downarrow Y = (1,\ 0.6,\ 0.5,\ 0.2)$$

· ***Qualitative Reasoning*** :

~ A mode of reasoning in which the input-output relation of a system is expressed as a collection of fuzzy IF-THEN fuzzy or linguistic variables.

~ For example, if X and Y are input variables and Z is the output variable, the relation between X,Y,and Z may be expressed as

$$\left.\begin{array}{l} \text{If } X \text{ is } A_1 \text{ and } Y \text{ is } B_1 \text{ then } Z \text{ is } C_1 \text{ .} \\ \text{If } X \text{ is } A_2 \text{ and } Y \text{ is } B_2 \text{ then } Z \text{ is } C_2 \text{ .} \\ \qquad\qquad \vdots \\ \text{If } X \text{ is } A_n \text{ and } Y \text{ is } B_n \text{ then } Z \text{ is } C_n \text{ .} \end{array}\right\} \quad (*)$$

where A_i, B_i and C_i, $i = 1,....,n$ are fuzzy subsets of their respective universe of discourse.

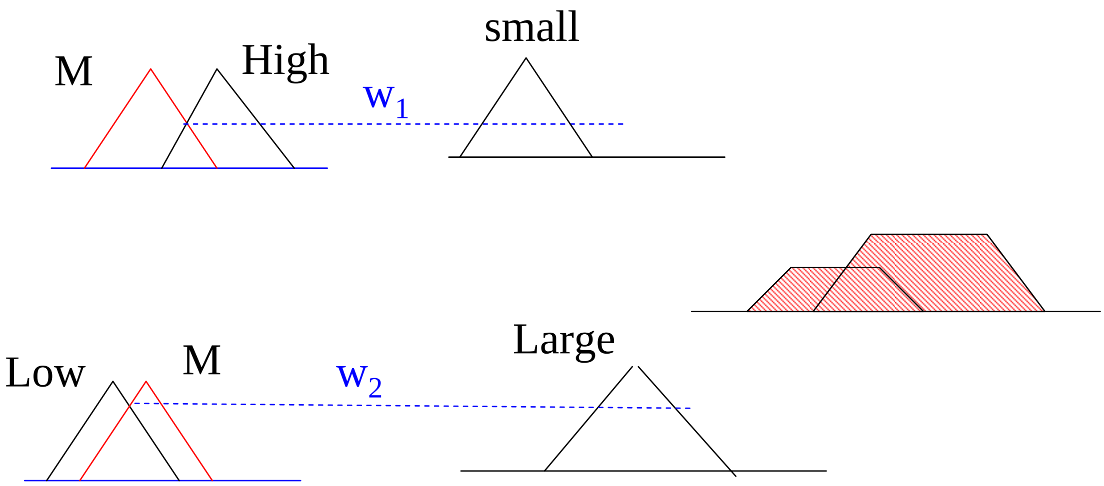
~ Given the dependence of Z on X and Y in the form of (*),we can employ the compositional rule of inference to compute the value of Z given the values of X and Y.

Ex : If pressure is High, Then volume is Small.

If pressure is Low, Then volume is Large.

If pressure is Medium, Then volume is max $\{w_1 \wedge \text{small}, w_2 \wedge \text{large}\}$

where $w_1 = \sup(\text{High} \wedge \text{Medium})$, $w_2 = \sup(\text{Low} \wedge \text{Medium})$, and can be considered as weighting coefficients that represent, respectively, the degrees to which the preconditions “High” and “Low” match the input “Medium.”



small
M
High
w1
Low
M
w2
Large