Chapter 3. Fuzzy Arithmetic

• ***Fuzzy arithmetic:***

~Addition(+) and subtraction (-):

Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ in $\Re$

If $x \in [a_1, a_2]$ and $y \in [b_1, b_2]$ than $x+y \in [a_1+b_1, a_2+b_2]$

Symbolically,we write

$$A(+)B = [a_1, a_2](+)[b_1, b_2] = [a_1+b_1, a_2+b_2]$$

subtraction:

$$A(-)B = [a_1, a_2](-) [b_1, b_2] = [a1-b_2, a_2-b_1]$$

~Image $\bar{A}$

If $x \in [a_1, a_2]$,then its image $-x \in [-a_2, -a_1]$

That is, if $A=[a_1, a_2]$ then its image is $\bar{A}=[-a_2, -a_1]$

Note that:

$$A(+)\bar{A} = [a_1, a_2](+)[-a_2, -a_1] = [a_1-a_2, a_2-a_1] \neq 0$$

$$A(-)B = A(+)\bar{B}=[a_1,a_2](+)[-b_2, -b_1] = [a_1-b_2, a_2-b_1]$$

~Multiplication(•) and division (:):

① Consider $A = [a_1, a_2]$ and $B = [b_1, b_2]$ in $\Re^+$(nonnegative real line)

$$A(\bullet)B = [a_1,a_2](\bullet)[b_1, b_2] = [a_1 \bullet b_1, a_2 \bullet b_2]$$

② If $A, B \subset \Re$, then there are nine possible combinations

③ If $k \in \Re^+$ and $A \subset \Re$, then

$$k \bullet A = [k, k](\bullet)[a_1, a_2] = [ka_1,ka_2]$$

④For the division of two intervals of confidence in $\Re_0^{\ +}$, we have

$$A(:)B = [a_1, a_2] (:)[b_1, b_2] = [a_1/b_2, a_2/b_1]$$

• *Inverse A^{-1} :*

If $x \in [a_1,a_2] \subset \Re_0^{\ +}$, where $\Re_0^{\ +}$ is the positive real line, then its inverse is $1/x \in [1/a_2, 1/a_1]$ and

$$A^{-1} = [a_1,a_2]^{-1} = [\frac{1}{a_2}, \frac{1}{a_1}]$$

$$A(:)B \equiv A(\bullet)B^{-1} = [a_1,a_2](\bullet)[\frac{1}{b_2}, \frac{1}{b_1}] = [\frac{a_1}{b_2}, \frac{a_2}{b_1}]$$

for $k > 0$

$$A(:)k \equiv A(\bullet)[\frac{1}{k}, \frac{1}{k}] = [\frac{a_1}{k}, \frac{a_2}{k}]$$

• *Max ($\vee$) and min ($\wedge$) operations:*

$$A(\vee)B = [a_1, a_2](\vee)[b_1, b_2] = [a_1 \vee b_1, a_2 \vee b_2]$$

$$A(\wedge)B = [a_1, a_2](\wedge)[b_1, b_2] = [a_1 \wedge b_1, a_2 \wedge b_2]$$

•Algebraic properties of addition and multiplication on intervals

Property	Addition(+)	Multiplication(•)
Intervals	For all A, B, C $\subset \Re$	For all A,B,C$\subset \Re^+$
Commutativity	$A(+)B = B(+)A$	$A(\bullet)B = B(\bullet)A$
Associativity	$(A(+)B)(+)C = A(+)(B(+)C)$	$(A(\bullet)B)(\bullet)C = A(\bullet)(B(\bullet)C)$
Neutral number	$A(+)0 = 0(+)A = A$	$A(\bullet)1 = 1(\bullet)A = A$
Image and inverse	$A(+)\bar{A} = \bar{A}(+)A \neq 0$	$A(\bullet)A^{-1} = A^{-1}(\bullet)A \neq 1$

• Given two fuzzy numbers A and B in $\Re$, for a specific $\alpha_1 \in [0,1]$, we will obtain two closed intervals, $A_{\alpha 1} \triangleq [a_1^{(\alpha 1)}, a_2^{(\alpha 1)}]$ from fuzzy number A, and $B_{\alpha 1} \triangleq [b_1^{(\alpha 1)}, b_2^{(\alpha 1)}]$ from fuzzy number B. The interval arithmetic can be applied to these two closed intervals.

• Let (☆) denote an arithmetic operation on fuzzy numbers such as addition(+), subtraction (-), multiplication(•)or division(:).

Using the extension principle, the result A (☆) B, where A and B are two fuzzy numbers, can be obtained:

$$\mu_{A(☆)B}(z) = \sup_{z = x☆y} \min \{\mu_A(x), \mu_B(y)\}$$

Using the α -cuts, we have

$$(A\,(☆)B)_\alpha = A_\alpha (☆) B_\alpha \text{, for all } \alpha \in [0,1]$$

where $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$, and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$

• For two fuzzy numbers A and B in $\Re$, we have

$$A_\alpha (+) B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}]$$

$$A_\alpha (-) B_\alpha = [a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}]$$

• For two fuzzy numbers A and B in $\Re^+ = [0, \infty)$ we have

$$A_\alpha (\bullet) B_\alpha = [a_1^{(\alpha)} \bullet b_1^{(\alpha)}, a_2^{(\alpha)} \bullet b_2^{(\alpha)}]$$

$$A_\alpha (:) B_\alpha = [\frac{a_1^{(\alpha)}}{b_2^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}]$$

- The multiplication of a fuzzy number $A \subset \Re$ by an ordinary number $k \in \Re^+$:

$$(k \bullet A)_\alpha = k \bullet A_\alpha = [ka_1^{(\alpha)}, ka_2^{(\alpha)}]$$

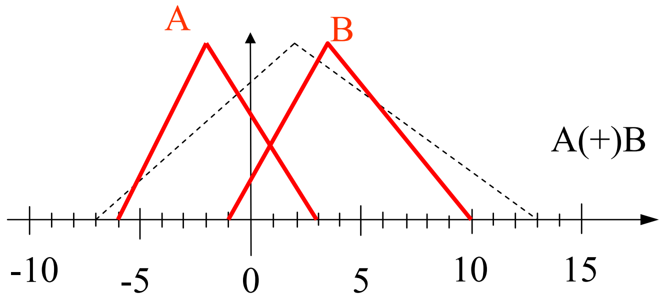
or,equivalently,

$$\mu_{k \bullet A}(x) = \mu_A(x/k), \quad \text{for all } x \in \Re$$

Ex 5.6:

Compute A(+)B and A(-)B,where

$$\mu_A(x)=\begin{cases} 0 & x \le -6 \\ (x+6)/4 & -6 < x \le -2 \\ (-x+3)/5 & -2 < x \le 3 \\ 0 & x > 3 \end{cases}, \quad \mu_B(x)=\begin{cases} 0 & x \le -1 \\ (x+1)/5 & -1 < x \le 4 \\ (-x+10)/6 & 4 < x \le 10 \\ 0 & x > 10 \end{cases}$$



① Find α -cuts $[a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $[b_1^{(\alpha)}, b_2^{(\alpha)}]$

$$\frac{a_1^{(\alpha)} + 6}{4} = \alpha, \quad \text{and} \quad \frac{-a_2^{(\alpha)} + 3}{5} = \alpha$$

Solving $a_1^{(\alpha)}$ and $a_2^{(\alpha)}$, we obtain

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [4\alpha - 6, -5\alpha + 3]$$

Similarly, $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [5\alpha - 1, -6\alpha + 10]$

② $A_\alpha (+) B_\alpha = [9\alpha - 7, -11\alpha + 13] \triangleq [c_1^{(\alpha)}, c_2^{(\alpha)}] = (A(+)B)_\alpha$

③ $x = 9\alpha - 7, \quad \alpha = \dfrac{x+7}{9}$

$x = -11\alpha + 13, \quad \alpha = \dfrac{-x+13}{11}$

$$\Rightarrow \mu_{A(+)B}(x)=\begin{cases} 0 & x \le -7 \\ (x+7)/9 & -7 < x \le 2 \\ (-x+13)/11 & 2 < x \le 13 \\ 0 & x > 13 \end{cases}$$

A_α (-) B_α = [10α - 16,-10α + 4]=$(A(-)B)_\alpha$

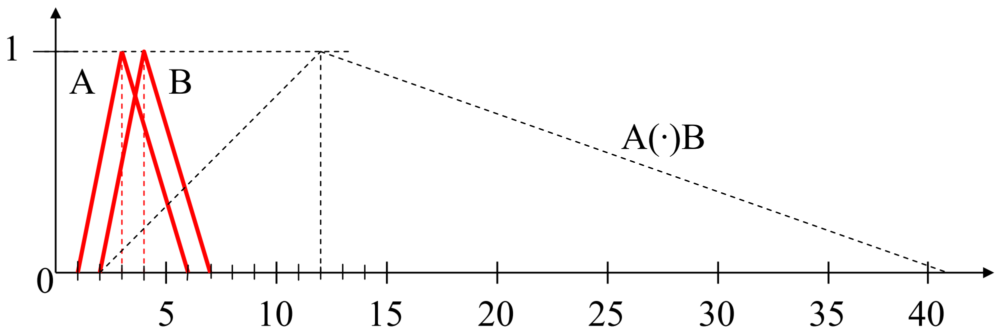
$10\alpha - 16 = x$, $\alpha = \dfrac{x+16}{10}$

$-10\alpha + 4 = x$, $\alpha = \dfrac{4-x}{10}$

$$\mu_{A(-)B}(x) = \begin{cases} 0 & x \le -16 \\ \dfrac{x+16}{10} & -16 < x \le -6 \\ \dfrac{-x+4}{10} & -6 < x \le 4 \\ 0 & x > 4 \end{cases}$$

Ex 5.7: Compute A(·)B, A(:)B

$$\mu_A(x) = \begin{cases} 0 & x \le 1 \\ \dfrac{x-1}{2} & 1 < x \le 3 \\ \dfrac{-x+6}{3} & 3 < x \le 6 \\ 0 & x > 6 \end{cases} \qquad \mu_B(x) = \begin{cases} 0 & x \le 2 \\ \dfrac{x-2}{2} & 2 < x \le 4 \\ \dfrac{-x+7}{3} & 4 < x \le 7 \\ 0 & x > 7 \end{cases}$$



compute α-cuts

$\alpha = \dfrac{a_1^{(\alpha)} - 1}{2}$ $\qquad a_1^{(\alpha)} = 2\alpha + 1$

$\alpha = \dfrac{-a_2^{(\alpha)} + 6}{3}$ $\qquad a_2^{(\alpha)} = -3\alpha + 6$

$A_\alpha = [\, a_1^{(\alpha)},\ a_2^{(\alpha)} \,] = [2\alpha + 1, -3\alpha + 6]$

similarly, from $\mu_B(x)$, we have $B_\alpha = [2\alpha + 2, -3\alpha + 7]$

$A_\alpha (\cdot) B_\alpha = [(2\alpha + 1)(2\alpha + 2), (-3\alpha + 6)(-3\alpha + 7)]$

$= [\, 4\alpha^2 + 6\alpha + 2, 9\alpha^2 - 39\alpha + 42] = (A(\cdot)B)_\alpha$

Solve $\quad 4\alpha^2 + 6\alpha + 2 = x$, $\alpha = (-3 \pm \sqrt{1+4x}\ )/4$

$\qquad 9\alpha^2 - 39\alpha + 42 = x$, $\alpha = (13 \pm \sqrt{1+4x}\ )/6$

$\because \alpha \in [0,1]$

$$\therefore \mu_{A(\cdot)B}(x)=\begin{cases} 0 & x \le 2 \\ (-3+\sqrt{1+4x})/4 & 2<x \le 12 \\ (13-\sqrt{1+4x})/6 & 12<x \le 42 \\ 0 & x>42 \end{cases}$$

For $A_\alpha (:) B_\alpha = [\dfrac{2\alpha+1}{-3\alpha+7} , \dfrac{-3\alpha+6}{2\alpha+2}]$

$$\Rightarrow \mu_{A(:)B}(x)=\begin{cases} 0 & x \le \frac{1}{7} \\ \frac{7x-1}{3x+2} & \frac{1}{7}< x \le \frac{3}{4} \\ \frac{-2x+6}{2x+3} & \frac{3}{4}< x \le 3 \\ 0 & x>3 \end{cases}$$

~k(·)A, k=5

$$\Rightarrow \mu_{k(\cdot)A}(x)=\mu_A(\frac{x}{5})=\begin{cases} 0 & x \le 5 \\ \frac{x-5}{10} & 5<x \le 15 \\ \frac{-x+30}{15} & 15<x \le 30 \\ 0 & x>30 \end{cases}$$

- Properties of operations on fuzzy numbers:

1.If A and B are fuzzy numbers in $\Re$, then A(+)B and A(-)B are also fuzzy numbers

2.If A and B are fuzzy numbers in $\Re^+$, then A(·)B and A(:)B are also fuzzy numbers

3.There are no image and inverse fuzzy numbers $\bar{A}$ and A^{-1}, respectively, such that

$A(+)\bar{A}=0$ and $A(\cdot) A^{-1}=1$

4. $(A(-)B)(+)B \neq A$ and $(A(:)B)(\cdot)B \neq A$

To prove the above properties, the concept of α -cuts is usually used

Thm 5-1:

Let A,B,and C be fuzzy numbers in $\Re^+$; then the following distributives holds

$$(A(+)B)(\cdot) C = (A (\cdot) C)(+)(B (\cdot)C)$$

pf:

$$\begin{aligned}((A(+)B)(\cdot) C)_\alpha &= (A_\alpha(+)B_\alpha)(\cdot)C_\alpha \\ &= ([a_1^{(\alpha)}, a_2^{(\alpha)}] (+) [b_1^{(\alpha)}, b_2^{(\alpha)}])(\cdot) [c_1^{(\alpha)}, c_2^{(\alpha)}] \\ &= [a_1^{(\alpha)}+b_1^{(\alpha)}, a_2^{(\alpha)}+b_2^{(\alpha)}] (\cdot) [c_1^{(\alpha)}, c_2^{(\alpha)}] \\ &= [a_1^{(\alpha)}\cdot c_1^{(\alpha)}+b_1^{(\alpha)}\cdot c_1^{(\alpha)}, a_2^{(\alpha)}\cdot c_2^{(\alpha)}+ b_2^{(\alpha)}\cdot c_2^{(\alpha)}]\end{aligned}$$

and $((A(\cdot)C)(+)(B(\cdot)C))_\alpha=([a_1^{(\alpha)}, a_2^{(\alpha)}](\cdot)[c_1^{(\alpha)},c_2^{(\alpha)}])(+)$
$([b_1^{(\alpha)},b_2^{(\alpha)}](\cdot)[c_1^{(\alpha)}, c_2^{(\alpha)}])$
$=[a_1^{(\alpha)}\cdot c_1^{(\alpha)}+b_1^{(\alpha)}\cdot c_1^{(\alpha)}, a_2^{(\alpha)}\cdot c_2^{(\alpha)}+b_2^{(\alpha)}\cdot c_2^{(\alpha)}]$

Remark: $(A(\cdot)B)(+)C \neq (A(+)C)(\cdot)(B(+)C)$

- ***Fuzzy minimum and fuzzy maximum*:**

Two fuzzy numbers A and B in $\Re$ are comparable, if

$$a_1^{(\alpha)}\le b_1^{(\alpha)} \text{ and } a_2^{(\alpha)}\le b_2^{(\alpha)} \quad \text{for all } \alpha\in[0.1]$$

and we can write $A \le B$

The fuzzy minimum and fuzzy maximum of two fuzzy numbers A and B in $\Re$ are denoted as $A(\wedge)B$ and $A(\vee)B$ and are defined, respectively, by

$$A_\alpha(\wedge)B_\alpha \equiv[\min(a_1^{(\alpha)},b_1^{(\alpha)}),\min(a_2^{(\alpha)},b_2^{(\alpha)})]=[a_1^{(\alpha)}\wedge b_1^{(\alpha)},a_2^{(\alpha)}\wedge b_2^{(\alpha)}]$$

$$A_\alpha(\vee)B_\alpha \equiv[\max(a_1^{(\alpha)},b_1^{(\alpha)}),\max(a_2^{(\alpha)},b_2^{(\alpha)})]=[a_1^{(\alpha)}\vee b_1^{(\alpha)},a_2^{(\alpha)}\vee b_2^{(\alpha)}]$$

Ex 5.8:

Consider two fuzzy numbers A and B with

$$\mu_A(x)=\begin{cases}0 & x\le -3\\ x+3 & -3<x\le -2\\ (-x+5)/7 & -2<x\le 5\\ 0 & x>5\end{cases} \quad \text{and} \quad \mu_B(x)=\begin{cases}0 & x\le -4\\ (x+4)/5 & -4<x\le 1\\ -x+2 & 1<x\le 2\\ 0 & x>2\end{cases}$$

$$\alpha = a_1^{(\alpha)}+3 \qquad a_1^{(\alpha)}=\alpha-3 \Rightarrow A_\alpha=[\alpha-3,-7\alpha+5]$$

$$\alpha=(-a_2^{(\alpha)}+5)/7 \quad a_2^{(\alpha)}=-7\alpha+5$$

$$B_\alpha=[5\alpha-4,-\alpha+2]$$

$$A_\alpha(\wedge)B_\alpha=[(\alpha-3)\wedge(5\alpha-4),(-7\alpha+5)\wedge(-\alpha+2)]\equiv[c_1^{(\alpha)}, c_2^{(\alpha)}]$$

there may be four combinations

$$[a_1^{(\alpha)}, a_2^{(\alpha)}], [a_1^{(\alpha)}, b_2^{(\alpha)}], [b_1^{(\alpha)}, a_2^{(\alpha)}], [b_1^{(\alpha)}, b_2^{(\alpha)}]$$

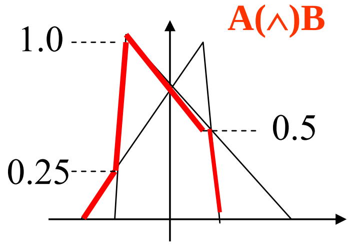
$$\alpha-3=5\alpha-4,\ \alpha=0.25;\quad -7\alpha+5=-\alpha+2,\ \alpha=0.5$$

$$\Rightarrow A_\alpha(\wedge)B_\alpha=\begin{cases}[5\alpha-4,-\alpha+2] & 0\le\alpha\le 0.25\\ [\alpha-3,-\alpha+2] & 0.25<\alpha\le 0.5\\ [\alpha-3,-7\alpha+5] & 0.5<\alpha\le 1\end{cases}$$

$$x = 5\alpha - 4, \alpha = \frac{x+4}{5}, \alpha = 0, x = -4, \alpha = 0.25, x = -2.75$$

$$x = -\alpha + 2, \alpha = -x + 2, \quad \alpha = 0, x = 2, \alpha = 0.5, x = 1.5$$

$$\mu_{A(\wedge)B}(x) = \begin{cases} 0 & x \le -4 \\ \frac{x+4}{5} & -4 < x \le -2.75 \\ x+3 & -2.75 < x \le -2 \\ \frac{-x+5}{7} & -2 < x \le 1.5 \\ -x+2 & 1.5 < x \le 2 \\ 0 & x > 2 \end{cases}$$



Similarly,

$$A_\alpha (\vee) B_\alpha = [(\alpha - 3) \vee (5\alpha - 4), (-7\alpha + 5) \vee (-\alpha + 2)] \equiv [c_1^{(\alpha)}, c_2^{(\alpha)}]$$

$$A_\alpha (\vee) B_\alpha = \begin{cases} [\alpha - 3, -7\alpha + 5] & 0 \le \alpha \le 0.25 \\ [5\alpha - 4, -7\alpha + 5] & 0.25 < \alpha \le 0.5 \\ [5\alpha - 4, -\alpha + 2] & 0.5 < \alpha \le 1 \end{cases}$$

$$\mu_{A(\vee)B}(x) = \begin{cases} 0 & x \le -3 \\ x+3 & -3 < x \le -2.75 \\ \frac{x+4}{5} & -2.75 < x \le 1 \\ -x+2 & 1 < x \le 1.5 \\ \frac{-x+5}{7} & 1.5 < x \le 5 \\ 0 & x > 5 \end{cases}$$

- ***L-R fuzzy number:***

~ to increase computational efficiency

~parametric function

~ In general, a L-R reference (or shape) function, $\phi(x) \in [0,1]$

$$\phi(x) = \begin{cases} L(x) & -\infty \le x < 0 \\ 1 & x = 0 \\ R(x) & 0 < x < \infty \end{cases} \qquad x \in \Re$$

$L(x)$ and $R(x)$ map $\Re^+ \rightarrow [0,1]$ and are monotonic (nonincreasing)

~L(x) (or R(x)) is a reference function iff

(1) L(x) = L(-x)

(2) L(0) = 1

(3) L(x) is nonincreasing on $[0,+\infty]$

~ϕ(x) is a normal convex function $\Rightarrow$ fuzzy number (LR)

Thus, L-R fuzzy number is represented by a pair of functions L&R with non-increasing monotonicity

Ex: $L(x) = \max(0,1- x^p)$, x>=0, p>0, L(x)=L(-x)

$L(x) = e^{-x}$, x>=0, L(x)=L(-x)

$L(x) = e^{-x^2}$

Def: A fuzzy number M is called L-R fuzzy number iff

$$\mu_M(x) = \begin{cases} L\left(\dfrac{m-x}{\alpha}\right) & x \le m \\ 1 & x = m \\ R\left(\dfrac{x-m}{\beta}\right) & x \ge m \end{cases}$$

where m: mean of the fuzzy number ; α, β: left and right spreads, respectively

Remarks:

(1) when spreads are zero $\Rightarrow$ nonfuzzy number

(2) When spreads increase $\Rightarrow$ M becomes more fuzzy

(3) m < 0 we have a left translation

m > 0 we have a right translation

(4) If $\alpha < 1$& $\beta<1$, then we have a contraction

$\alpha > 1$ & $\beta>1$, then we have a dilation

Hence, a L-R fuzzy number can be represented by 3 parameters m, α, β

$M=(m, \alpha, \beta)_{LR}$

Ex:Consider $L(x)=\dfrac{1}{1+x^2}$, $R(x)=\dfrac{1}{1+2|x|}$, $m=5,\alpha=2,\beta=3$

$$\mu_M(x) = \begin{cases} L\left(\dfrac{m-x}{\alpha}\right)=L\left(\dfrac{5-x}{2}\right)=\dfrac{1}{1+\left(\dfrac{5-x}{2}\right)^2} & x<5 \\ 1 & x=5 \\ R\left(\dfrac{x-m}{\beta}\right)=R\left(\dfrac{x-5}{3}\right)=\dfrac{1}{1+2\left|\dfrac{x-5}{3}\right|} & x>5 \end{cases}$$

~Consider $A=(m, \alpha, \beta)_{LR}$, $B=(n, r, \delta)_{LR}$ in $\Re$ with nonincreasing L-R fuzzy number

- Addition: $A(+)B = (m, \alpha, \beta)_{LR} (+) (n, r, \delta)_{LR} = (m+n, \alpha+r, \beta+\delta)_{LR}$
- Image: $\bar{A} = -(m, \alpha, \beta)_{LR} = (-m, \beta, \alpha)_{RL}$
- Subtraction: $A=(m, \alpha, \beta)_{LR}$, $B=(n, r, \delta)_{RL}$ in $\Re$

  $A(-)B = (m, \alpha, \beta)_{LR} (-) (n, r, \delta)_{RL} = (m-n, \alpha+\delta, \beta+r)_{LR}$

- Multiplication:

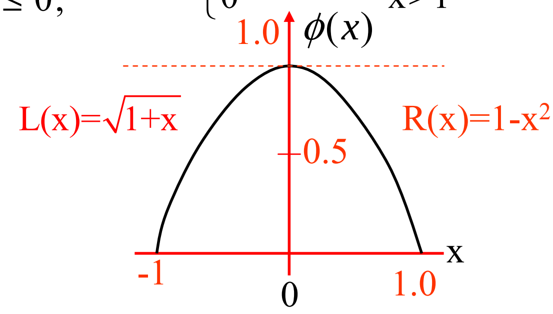
$$k \in \Re,\ k(\bullet)A = k(\bullet)(m, \alpha, \beta)_{LR} = \begin{cases} (km, k\alpha, k\beta)_{LR} & k>0 \\ (km, -k\beta, -k\alpha)_{RL} & k<0 \end{cases}$$

Ex: $L(x)=\begin{cases} 0 & x<-1 \\ \sqrt{1+x} & -1 \le x \le 0, \end{cases}$ $\quad R(x)=\begin{cases} 1-x^2 & 0 \le x \le 1 \\ 0 & x>1 \end{cases}$

$A = (m, \alpha, \beta)_{LR} = (3,1,3)_{LR}$

$B = (n, r, \delta)_{LR} = (6,2,4)_{LR}$

$C = (p, \zeta, \lambda)_{RL} = (8,4,5)_{RL}$



$$L(x)=\begin{cases} \sqrt{1-x} & 0 \le x \le 1 \\ 0 & x \ge 1 \end{cases}$$

$$\mu_A(x) = \begin{cases} 0 & x \le 2 \\ \sqrt{1-(3-x)} & 2 \le x \le 3 \\ 1 & x=3 \\ 1-(\frac{x-3}{3})^2 & 3 \le x \le 6 \\ 0 & 6 \le x \end{cases}, \quad \mu_C(x) = \begin{cases} 1-(\frac{8-x}{4})^2 & 4 \le x \le 8 \\ 1 & x=8 \\ \sqrt{1-(\frac{x-8}{5})} & 8 \le x \le 13 \\ 0 & \text{otherwise} \end{cases}$$

$A+B = (m+n, \alpha+r, \beta+\delta)_{LR} = (3+6, 1+2, 3+4)_{LR} = (9, 3, 7)_{LR}$

$A(-)C = (m, \alpha, \beta)_{LR} - (p, \zeta, \lambda)_{RL} = (m-p, \alpha+\lambda, \beta+\zeta)_{LR}$

$=(3-8, 1+5, 3+4)_{LR} = (-5, 6, 7)_{LR}$

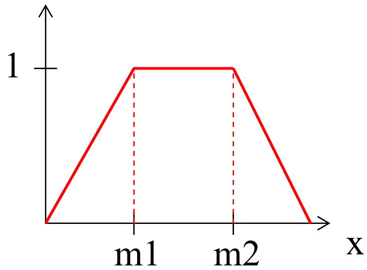
$M=(m, \alpha, \beta)_{LR}$

a point

~L-R fuzzy interval:

$M=(m_1, m_2, \alpha, \beta)_{LR}$

$$\mu_M(x) = \begin{cases} L\left(\dfrac{m_1 - x}{\alpha}\right) & x < m_1,\ \alpha > 0 \\ 1 & x \in [\, m_1, m_2] \\ R\left(\dfrac{x - m_2}{\beta}\right) & x > m_2,\ \beta > 0 \end{cases}$$



L and R: nonincreasing functions

- Consider two L-R fuzzy intervals $M = (m_1, m_2, \alpha, \beta)_{LR}$ and $N = (n_1, n_2, r, \delta)_{LR}$ in R with nonincreasing L and R reference functions

~Addition:

$$M(+)N = (m_1, m_2, \alpha, \beta)_{LR} (+) (n_1, n_2, r, \delta)_{LR}$$
$$= (m_1+n_1, m_2+n_2, \alpha + r, \beta + \delta)_{LR}$$

~Image:

$$\overline{M} = -(m_1, m_2, \alpha, \beta)_{LR} = (-m_2, -m_1, \beta, \alpha)_{RL}$$

~Subtraction:

$$M = (m_1, m_2, \alpha, \beta)_{LR} \text{ and } N = (n_1, n_2, r, \delta)_{RL}$$
$$M(-)N = (m_1, m_2, \alpha, \beta)_{LR} (-) (n_1, n_2, r, \delta)_{RL}$$
$$= (m_1 - n_2, m_2 - n_1, \alpha + \delta, \beta + r)_{LR}$$

~Multiplication:

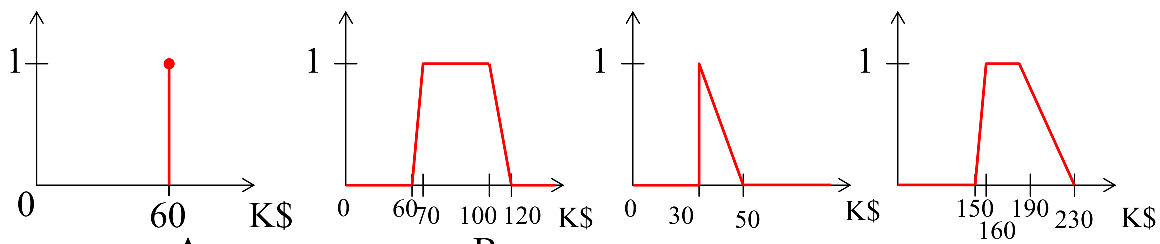
$$k \in R,\ k(\bullet)M = k(\bullet)(m_1, m_2, \alpha, \beta)_{LR}$$
$$= \begin{cases} (km_1, km_2, k\alpha, k\beta)_{LR} & k>0 \\ (km_2, km_1, |k|\beta, |k|\alpha)_{RL} & k<0 \end{cases}$$

Ex: Consider setting up a budget from three sources:

source A: A certain and precise amount of \$60k

source B: may range from \$60k to \$120k. But it is reasonable to except an amount of \$70k to 100k

source C: may amount to \$30k but not be more than \$50k



source A source B Source C total S

$L(x) = R(x) = \max(0, 1-x), \quad x \in \Re^+$

$A = (60, 60, 0, 0)_{LR}$, $B = (70, 100, 10, 20)_{LR}$, $C = (30, 30, 0, 20)_{LR}$

$S = A \,(+)\, B \,(+)\, C$

$= (60, 60, 0, 0)_{LR} \,(+)\, (70, 100, 10, 20)_{LR} \,(+)\, (30, 30, 0, 20)_{LR}$

$= (160, 190, 10, 40)_{LR}$

the possible amount is \$150,000 to \$230,000 and

the expected amount is \$160,000 to \$190,000