Chapter 2. Fuzzy Relations

- Traditional crisp relation : based on the concept that everything is either related or unrelated.
- Fuzzy relation : allows various degrees of interactions between elements.
- A crisp relation among crisp sets $X_1, X_2, \cdots, X_n$ is a crisp subset on the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$. This relation is denoted by $R(X_1, X_2, \cdots, X_n)$. Hence

  $$R(X_1, X_2, \cdots, X_n) \subset X_1 \times X_2 \times \cdots \times X_n.$$

  where $X_1 \times X_2 \times \cdots \times X_n = \{ (x_1, x_2, \cdots, x_n) \mid x_i \in X_i, i \in \{1, \cdots, n\}\}$.

  It's interpreted that relation R exists among $\{X_1, X_2, \cdots, X_n\}$ if the tuple $(x_1, x_2, \cdots, x_n)$ is in the set $R(X_1, \cdots, X_n)$.

- *Fuzzy relation :*
  A fuzzy set defined on the Cartesian product of crisp sets $\{X_1, X_2, \ldots, X_n\}$, where tuples $(x_1, x_2, \ldots, x_n)$ may have varying degrees of membership $\mu_R(x_1, x_2, \ldots, x_n)$ within the relation. That is :

$$R(X_1, X_2, \cdots, X_n) = \int_{x_1 \cdots x_n} \mu_R(x_1, x_2, \cdots x_n) / (x_1, x_2, \cdots x_n), \quad x_i \in X_i$$

e. g. Consider two crisp sets X_1 and X_2. Then
$R(X_1, X_2) = \{ ((x_1, x_2), \mu_R(x_1, x_2)) \mid (x_1, x_2) \in X_1 \times X_2 \}$
is a fuzzy relation on $X_1 \times X_2$.

Ex:
Let $X = \{x_1, x_2\}$ = { New York City (NYC) , Tokyo (TKO) }
$Y = \{y_1, y_2, y_3\}$
= { Taipei (TPE) , Hong Kong (HKG) , Bejing (BJI) }.
R : “very close”

Crisp relation : may be defined by the following $\mu_R(x_i, y_i)$

		y_1	y_2	y_3
		TPE	HKG	BJI
x_1	NYC	0	0	0
x_2	TKO	1	1	1

Fuzzy relation :

		y_1	y_2	y_3
		TPE	HKG	BJI
x_1	NYC	0.3	0.1	0.1
x_2	TKO	1	0.7	0.8

$$\therefore \quad \mathrm{R}(X,Y) = 0.3/(NYC,TPE) + 0.1/(NYC,HKG) + 0.1/(NYC,BJI)$$
$$+ 1/(TKO,TPE) + 0.7/(TKO,HKG) + 0.8/(TKO,BJI)$$

Ex : $X = Y = \mathbf{R}^1$, $x \in X$, $y \in Y$.

Fuzzy relation R = "y is much larger than x" on $X \times Y$.

$$\mu_R(x, y) = \begin{cases} 0 & x \geq y \\ \dfrac{1}{1+[5/(y-x)^2]} & x < y \end{cases}$$

- *Binary (fuzzy) relation*

A fuzzy relation between two sets X and Y and is denoted by R(X ,Y).

Let $X = \{x_1, x_2, \cdots x_n\}$ and $Y = \{y_1, y_2 \cdots y_m\}$. R (X , Y) can be expressed

by an $n \times m$ matrix as :

$$R(X,Y) = \begin{bmatrix} \mu_R(x_1,y_1) & \mu_R(x_1,y_2) & \cdots & \mu_R(x_1,y_m) \\ \mu_R(x_2,y_1) & \mu_R(x_2,y_2) & \cdots & \mu_R(x_2,y_m) \\ \vdots & & & \\ \mu_R(x_n,y_1) & \mu_R(x_n,y_2) & \cdots & \mu_R(x_n,y_m) \end{bmatrix}$$

Such a matrix is called a fuzzy matrix

- *Domain of a binary fuzzy relation R(X , Y):*

  the fuzzy set “ dom R(X ,Y) ” with

$$\mu_{dom R}(x) = \max_{y \in Y} \mu_R(x, y) \quad \text{for each } x \in X$$

- *Range of a binary fuzzy relation R(X , Y):*

  the fuzzy set “ ran R(X,Y)” with

$$\mu_{ran R}(y) = \max_{x \in X} \mu_R(x, y) \quad \text{for each } y \in Y$$

- *Height:*

$$H(R) = \sup_{y \in Y} \sup_{x \in X} \mu_R(x, y).$$

  if H(R) =1 , R is a normal fuzzy relation .

- *Resolution form :*

  every binary fuzzy relation R (X , Y) can be represented by

$$R = \bigcup_{\alpha \in \Lambda_R} \alpha \mathrm{R}_\alpha$$

Ex :

$$R = \begin{bmatrix} 0.4 & 0.5 & 0 \\ 0.9 & 0.5 & 0 \\ 0 & 0 & 0.3 \\ 0.3 & 0.9 & 0.4 \end{bmatrix}$$

R = 0.3 $R_{0.3}$ + 0.4 $R_{0.4}$ + 0.5$R_{0.5}$ + 0.9 $R_{0.9}$

$$R = 0.3 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} + 0.4 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} + 0.5 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + 0.9 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- ***Operations on fuzzy relations :***

  ~ *Projection* :

  Given a fuzzy relation R(X , Y) . Let [R ↓ Y] denote the projection of R on to Y. Then [R ↓ Y] is a fuzzy set (relation) in Y, whose membership function is defined by

$$\mu_{[R \downarrow Y]}(y) = \max_{x} \mu_R(x, y)$$

Remark : the max operator may be replaced with any t - conorm.

Ex3.5 : Consider

$$R(X,Y)= \begin{array}{c c} & \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{pmatrix} 0.2 & 0.5 & 1 & 0 & 0.6 \\ 0.1 & 0 & 0.7 & 0.4 & 0 \\ 0.9 & 0.2 & 0 & 0.2 & 1 \end{pmatrix} \end{array}$$

$[R \downarrow X] = 1/x_1 + 0.7/x_2 + 1/x_3$

$[R \downarrow Y] = 0.9/y_1 + 0.5/y_2 + 1/y_3 + 0.4/y_4 + 1/y_5$

Let $X_1 = \{ a , b \}$, $X_2 = \{ c , d \}$, $X_3 = \{f , g\}$.

$R (X_1 , X_2 , X_3) = 0.3/acf + 0.7/adf + 0.1/bcg + 0.8/bdf + 1/bdg$.

$R_{1,2} \equiv [R \downarrow \{ X_1 , X_2 \}] = 0.3/ac + 0.7/ad + 0.1/bc + 1/bd$.
$R_{1,3} \equiv [R \downarrow \{ X_1 , X_3 \}] = 0.7/af + 0.8/bf + 1/bg$.
$R_{2,3} \equiv [R \downarrow \{ X_2 , X_3 \}] = 0.3/cf + 0.1/cg + 0.8/df + 1/dg$.

$R_1 \equiv [R \downarrow X_1] = 0.7/a + 1/b$.

$R_2 \equiv [R \downarrow X_2] = 0.3/c + 1/d$.

$R_3 \equiv [R \downarrow X_3] = 0.8/f + 1/g$.

- *Cylindric extension :*

Given a fuzzy relation R (X) or a fuzzy set R on X, let $[R \uparrow Y]$ denote the cylindric extension of R into Y. Then $[R \uparrow Y]$ is a fuzzy relation in $X \times Y$ with $\mu_{[R \uparrow Y]} (x , y) = \mu_R (x)$, $x \in X , y \in Y$.

Equivalently, assume $X \times Y$ is the whole Cartesian product space.

Let R be a fuzzy set on X, where $X = \{ x_1, x_2, \cdot \cdot x_n \}$,

then $[R \uparrow Y] = R \times Y$,

where we consider that $Y = 1/y_1 + 1/y_2 + \cdot \cdot \cdot + 1/y_n$.

Similarly, if R is a fuzzy set on Y , then

$$[\, R \uparrow X\,] = X \times R\,.$$

e.g. $X = \{ x_1, x_2 \}$, $Y = \{ y_1, y_2 \}$. $R = \mu_R(x_1)/x_1 + \mu_R(x_2)/x_2$, and

$Y = 1/y_1 + 1/y_2$, then

$$[\, R \uparrow Y\,] = R \times Y = \mu_R(x_1)/(x_1 , y_1) + \mu_R(x_1)/(x_1 , y_2) + \mu_R(x_2)/(x_2,y_1) + \mu_R(x_2)/(x_2,y_2).$$

~ the concept of cylindric extension can be used to extend an r – ary relation $R (X_1, \cdots ,X_r)$ to an n –ary relation $R(X_1, X_2, \cdots ,X_r , \cdots ,X_n)$ where $n > r$.

Ex 3.6 :

$$[[R \downarrow X] \uparrow Y] = \begin{array}{cc} \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} & \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} & \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \end{array}$$

$[R \downarrow X] = 1/x_1 + 0.7/x_2 + 1/x_3$

$$[[R \downarrow Y] \uparrow X] = \begin{bmatrix} 0.9 & 0.5 & 1 & 0.4 & 1 \\ 0.9 & 0.5 & 1 & 0.4 & 1 \\ 0.9 & 0.5 & 1 & 0.4 & 1 \end{bmatrix}$$

$[R \downarrow Y] = 0.9/y1 + 0.5/y2 + 1/y3 + 0.4/y4 + 1/y5$

~ Observation :

Fuzzy relations that can be reconstructed from one of their projections by cylindric extension are rather rare.

Solution :

reconstruct the fuzzy relation from several of its projections .

• *Cylindric closure*:

Let $Y = Y_1 \times Y_2 \times \cdots \times Y_n$ be the whole Cartesian product space, where Y_i may be more than one dimension. Given a set of projections of a fuzzy relation

$$\{ R_i \mid R_i \text{ is a projection on } Y_i , i \in [1 , n] \} .$$

The cylindric closure of these projections , denoted as cyl $\{R_i\}$ is defined by

cyl $\{ R_i \}$ =

$$[\, R_1 \uparrow (Y - Y_1)\,] \cap [\, R_2 \uparrow (Y - Y_2)\,] \cap \cdots \cap [\, R_n \uparrow (Y - Y_n)] .$$

where $(Y - Y_i)$ is the Cartesian product space without Y_i ,

i.e. $Y_1 \times Y_2 \times \cdots Y_{i-1} \times Y_{i+1} \times \cdots Y_n$, and $\cap$: t – norm operator.

Ex3.7: (from Ex 3.5)

_	$[R_{1,2} \uparrow X_3]$	$[R_{1,3} \uparrow X_2]$	$[R_{2,3} \uparrow X_1]$
(a , c , f)	0.3	0.7	0.3
(a , c , g)	0.3	0	0.1
(a , d , f)	0.7	0.7	0.8
(a , d , g)	0.7	0	1
(b , c , f)	0.1	0.8	0.3
(b , c , g)	0.1	1	0.1
(b , d , f)	1	0.8	0.8
(b , d , g)	1	1	1

cyl $\{ R_{1,2} , R_{1,3} , R_{2,3} \} = [R_{1,2} \uparrow X_3] \cap [R_{1,3} \uparrow X_2] \cap [R_{2,3} \uparrow X_1]$

$= 0.3/acf + 0.7/adf + 0.1/bcf + 0.1/bcg + 0.8/bdf + 1/bdg$.

~ Let $R_x \triangleq [R \downarrow X]$ and $R_Y \triangleq [R \downarrow Y]$, and then .

cyl $\{ R_X , R_Y \} = [R_X \uparrow Y] \cap [R_Y \uparrow X] = [R_X \times Y] \cap [X \times R_Y]$

$= R_X \times R_Y$.

Remark : $R(X , Y) \subseteq R_X \times R_Y$.

Ex 3.8 : (from Ex 3.6)

$R_X \times R_Y = [[R \downarrow X] \uparrow Y] \cap [[R \downarrow Y] \uparrow X]$

$$= \begin{bmatrix} 0.9 & 0.5 & 1 & 0.4 & 1 \\ 0.7 & 0.5 & 0.7 & 0.4 & 0.7 \\ 0.9 & 0.5 & 1 & 0.4 & 1 \end{bmatrix}$$

~ For a fuzzy relation R(X , Y) , the inverse fuzzy relation $R^{-1}(Y , X)$ is a relation defined on $Y \times X$ by

$$\mu_{R^{-1}}(y, x) \triangleq \mu_R(x,y) , \text{ for all } (x,y) \in X \times Y .$$

Ex :

X = { x_1 , x_2 , x_3 } , Y = { y_1 , y_2 }

$$R(X, Y) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\!\!\begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{pmatrix} 0.2 & 1 \\ 0 & 0.4 \\ 0.8 & 0 \end{pmatrix} \end{array} \qquad R^{-1}(Y, X) = \begin{array}{c} \\ y_1 \\ y_2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 0.2 & 0 & 0.8 \\ 1 & 0.4 & 0 \end{pmatrix} \end{array}$$

• *Composition of fuzzy relations*

~ there are two types of composition operators :

max–min and *min–max* compositions .

~ these compositions can be applied to both *relation–relation* compositions and *set– relation* compositions.

~ *relation–relation* compositions:

1. Let P(X,Y) and Q(Y,Z) be two fuzzy relations on X×Y and Y×Z . The max –min composition of P(X,Y) and Q(Y,Z) , denoted as P(X,Y) ∘ Q(Y,Z) is defined by

$$\mu_{P\circ Q}(x,z) \triangleq \max_{y\in Y} \min [\, \mu_P(x,y) , \mu_Q(y,z)] \,.\; x \in X , z \in Z$$

2. The min – max composition of P(X,Y) and Q(Y,Z) , denoted as P(X,Y) ▫ Q(Y,Z) is defined by

$$\mu_{P\,\square\, Q}(x,z) \triangleq \min_{y\in Y} \max [\mu_P(x,y) , \mu_Q(y,z)].\; x\in X,\ z\in Z .$$

Ex :

X = {x_1 , x_2 }. Y = {y_1 , y_2 , y_3 } , Z = { z_1 , z_2 } .

$$P(X, Y) = \begin{array}{c} \\ x_1 \\ x_2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 0.3 & 0 & 0.7 \\ 0.8 & 1 & 0.4 \end{pmatrix} \end{array} \qquad Q(Y, Z) = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}\!\!\begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{pmatrix} 0.5 & 1 \\ 0 & 0.9 \\ 0.2 & 0.6 \end{pmatrix} \end{array}$$

$$P \circ Q(X, Z) = \begin{pmatrix} 0.3 & 0 & 0.7 \\ 0.8 & 1 & 0.4 \end{pmatrix} \circ \begin{pmatrix} 0.5 & 1 \\ 0 & 0.9 \\ 0.2 & 0.6 \end{pmatrix}$$

$$= \begin{pmatrix} (0.3\wedge0.5)\vee(0\wedge0)\vee(0.7\wedge0.2) & (0.3\wedge1)\vee(0\wedge0.9)\vee(0.7\wedge0.6) \\ (0.8\wedge0.5)\vee(1\wedge0)\vee(0.4\wedge0.2) & (0.8\wedge1)\vee(1\wedge0.9)\vee(0.4\wedge0.6) \end{pmatrix}$$

$$= \begin{array}{c c} & \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{pmatrix} 0.3 & 0.6 \\ 0.5 & 0.9 \end{pmatrix} \end{array}$$

Ex : X = { Peter , Mary , John }, Y = { y_1 , y_2 , y_3 , y_4 } = { theory , application , hardware , programming }, Z = { FT , FC , NN , ES } .

P(X,Y) : student's interest, Q(Y, Z) : course property.

$$P(X, Y) = \begin{array}{c c} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} Peter \\ Mary \\ Tohn \end{array} & \begin{pmatrix} 0.2 & 1 & 0.8 & 0.1 \\ 1 & 0.1 & 0 & 0.5 \\ 0.5 & 0.9 & 0.5 & 1 \end{pmatrix} \end{array} \qquad Q(Y, Z) = \begin{array}{c c} & \begin{array}{cccc} FT & FC & NN & ES \end{array} \\ \begin{array}{c} y_1 \\ y_2 \\ y_3 \\ y_4 \end{array} & \begin{pmatrix} 1 & 0.5 & 0.6 & 0.1 \\ 0.2 & 1 & 0.8 & 0.8 \\ 0 & 0.3 & 0.7 & 0 \\ 0.1 & 0.5 & 0.8 & 1 \end{pmatrix} \end{array}$$

$$P \circ Q = \begin{array}{c c} & \begin{array}{cccc} FT & FC & NN & ES \end{array} \\ \begin{array}{c} \text{Peter} \\ \text{Mary} \\ \text{John} \end{array} & \begin{pmatrix} 0.2 & 1 & 0.8 & 0.8 \\ 1 & 0.5 & 0.6 & 0.5 \\ 0.5 & 0.9 & 0.8 & 1 \end{pmatrix} \end{array}$$

~ *set–relation* composition :

1. Let A be a fuzzy set on X and R(X , Y) be a fuzzy relation on X$\times$Y. The max-min composition of A on R(X , Y) , denoted as A$\circ$R(X , Y) , is defined by

$$\mu_{A\circ R}(y) \triangleq \max_{x\in X} \min[\mu_A(x), \mu_R(x, y)] \text{ for all } y \in Y .$$

Fuzzy input A on X $\longrightarrow$ $\boxed{\begin{array}{c}\text{Fuzzy relation} \\ R(X, Y)\end{array}}$ $\longrightarrow$ Fuzzy output B . on Y

Ex :

X = Y = { 1, 2, 3, 4 }, A : " small ", A = 1/1 + 0.6/2 + 0.2/3

R: approximately equal.

$$A \circ R(X, Y) = (1\ \ 0.6\ \ 0.2\ \ 0) \circ \begin{array}{c|cccc|c} & y_1 & y_2 & y_3 & y_4 & \\ \hline & 1 & 0.5 & 0 & 0 & x_1 \\ & 0.5 & 1 & 0.5 & 0 & x_2 \\ & 0 & 0.5 & 1 & 0.5 & x_3 \\ & 0 & 0 & 0.5 & 1 & x_4 \end{array}$$

$$= (1\ \ 0.6\ \ 0.5\ \ 0.2).$$

$\therefore$ B = 1/1 + 0.6/2 + 0.5/3 + 0.2/4

2. The min – max composition of A and R(X , Y), denoted as A ▫ R(X , Y), is defined by

$$\mu_{A \square R}(y) \triangleq \min_{x \in X} \max\,[\mu_A(x), \mu_R(x, y)], \text{ for all } y \in Y.$$

- ***Various Type of Binary Fuzzy Relations***

~ *reflexive :*

A fuzzy relation R(X , X) is reflexive iff

$$\mu_R(x, x) = 1, \quad \text{for all } x \in X.$$

If this is not the case for some $x \in X$, then R(X , X) is irreflexive. If this is not the case for all $x \in X$, then the relation is called antireflexive .

~ *symmetric:*

A fuzzy relation R(X , X) is symmetric iff.

$$\mu_R(x,y) = \mu_R(y,x) \quad \text{for all } x, y \in X.$$

If it is not satisfied for some x, $y \in X$, then the relation is called asymmetric . If the equality is not satisfied for all member of the support of the relation, then it is called antisymmetric. If it is not satisfied for all $x, y \in X$, then R(X , X) is called strictly antisymmetric.

~ *transitive:*

A fuzzy relation R(X , X) is transitive (or more specifically , max – min transitive) iff

$$\mu_R(x,z) \geq \max_{y\in X} \min[\mu_R(x , y) , \mu_R(y , z)] . \text{ for all } (x , z) \in X^2 .$$

If this is not true for some members of X, R (X , X) is called nontransitive. If the inequality does not hold for all (x , z)∈ X^2 , then R(X , X) is called antitransitive.

generalization :

$$\mu_R(x,z) \geq \max_{y\in Y} t\,[\mu_R(x,y) , \mu_R(y,z)] .$$

e.g. max-product transitive

$$\mu_R(x,z) \geq \max_{y\in Y} [\mu_R(x,y)\bullet\mu_R(y,z)] . \quad (x,z)\in X^2$$

Ex :

$$R_a = \begin{bmatrix} 1 & 0.8 & 0.3 \\ 0.3 & 1 & 0.6 \\ 0.4 & 0 & 1 \end{bmatrix} ,$$

(reflexive)

$$R_b = \begin{bmatrix} 0.3 & 1 & 0.9 \\ 0 & 0.7 & 0.2 \\ 0.5 & 0 & 0.3 \end{bmatrix}$$

(antireflexive)

$$R_c = \begin{bmatrix} 1 & 0.5 & 0.7 \\ 0.5 & 0.3 & 0.1 \\ 0.7 & 0.1 & 0 \end{bmatrix} ,$$

(symmetric)

$$R_d = \begin{bmatrix} 0.1 & 0 & 0.7 \\ 0 & 1 & 0.2 \\ 0 & 0.3 & 0.2 \end{bmatrix}$$

(antisymmetric)

$$R_e = \begin{bmatrix} 1 & 0 & 0.6 \\ 0.1 & 0.3 & 0.8 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

(strictly antisymmetric)

$$R_f = \begin{bmatrix} 0.1 & 0.5 & 0.7 \\ 0 & 1 & 0.2 \\ 0 & 0.3 & 0.2 \end{bmatrix}$$

(transitive)

$$R_f = \begin{bmatrix} 0.1 & 0.5 & 0.7 \\ 0 & 1 & 0.2 \\ 0 & 0.3 & 0.2 \end{bmatrix} \supseteq \begin{bmatrix} 0.1 & 0.5 & 0.2 \\ 0 & 1 & 0.2 \\ 0 & 0.3 & 0.2 \end{bmatrix} = R_f \circ R_f$$

Ex : " x and y are very near"

is reflexive, symmetric and nontransitive.

" x is greatly smaller than y"

is anti reflexive, strictly antisymmetric, and transitive.

• *Transitive closure ($R_T(X , X)$)*

The transitive closure of a fuzzy relation R(X , X) is denoted by $R_T(X , X)$; it is a fuzzy relation that is transitive and contains R(X , X), and its elements have the smallest possible membership grades. When X has n elements, $R_T(X , X)$ can be obtained by

$$R_T(X , X) = R \cup R^2 \cup \cdots \cup R^n, \quad where \quad R^i = R^{i-1} \circ R \quad i \geq 2.$$

Ex : Let X = $\{x_1 , x_2 , x_3 \}$. and

$$R = \begin{bmatrix} 0.1 & 0.5 & 0.2 \\ 0.4 & 0.7 & 0.7 \\ 0 & 0.1 & 0.4 \end{bmatrix} \qquad R^2 = R \circ R = \begin{bmatrix} 0.4 & 0.5 & 0.5 \\ 0.4 & 0.7 & 0.7 \\ 0.1 & 0.1 & 0.4 \end{bmatrix},$$

$$R^3 = \begin{bmatrix} 0.4 & 0.5 & 0.5 \\ 0.4 & 0.7 & 0.7 \\ 0.1 & 0.1 & 0.4 \end{bmatrix}, \quad R_T(X , X) = R \cup R^2 \cup R^3 = \begin{bmatrix} 0.4 & 0.5 & 0.5 \\ 0.4 & 0.7 & 0.7 \\ 0.1 & 0.1 & 0.4 \end{bmatrix}$$

• *Fuzzy Relation Equations :*

Let A be a fuzzy set in X and R(X,Y) be a binary fuzzy relation in X×Y. The set - relation composition of A and R , $A \circ R$, results in a fuzzy set in Y. Let us denote the resulting fuzzy set as B. Then we have

$$A \circ R = B.$$

and

$$\mu_B(y) = \mu_{A \circ R}(y) \triangleq \max_{x \in X} \min[\mu_A(x), \mu_R(x,y)].$$

~ Basic problem :

give any two items of A , B and R, how to find the third ?

P1 : Given A and R , determine B .

Ex3.21:

$A = 0.2/x_1 + 0.8/x_2 + 1/x_3$. $R = \begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.7 & 1 & 0.4 \\ 0.5 & 0.9 & 0.6 \\ 0.2 & 0.6 & 0.3 \end{bmatrix} \end{array}$

$$B = A \circ R = (0.2\ 0.8\ 1) \circ \begin{pmatrix} 0.7 & 1 & 0.4 \\ 0.5 & 0.9 & 0.6 \\ 0.2 & 0.6 & 0.3 \end{pmatrix} = (0.5\ 0.8\ 0.6)$$

B = 0.5/y1 + 0.8/y2 + 0.6/y3

P2 : Given A and B, determine R.
P3 : Given R and B, determine A.

Thm 3.1:

P2 has solutions iff the height of the fuzzy set A is greater than or equal to the height of the fuzzy set B , that is

$$(*) \qquad \max_{x\in X} \mu_A(x) \geq \mu_B(y)\ , \quad \text{for all } y \in Y\ .$$

Pf : => if (*) does not hold, i.e. $\max_{x\in X} \mu_A(x) < \mu_B(y)$, for some $y \in Y$.

Then no $\mu_R(x,y)$ satisfies

$$\mu_B(y) = \mu_{A\circ R}(y) = \max_{x\in X} \min[\mu_A(x)\ , \mu_R(x\ , y)]\ .$$

<= let $x^* \in X$ and $\mu_A(x^*) = \max_{x\in X} \mu_A(x)$

define the membership function of R as

$$\begin{cases} \mu_R(x^*, y) = \mu_B(y) & \text{for all } y \in Y \\ \mu_R(x\ , y) = 0 & \text{for all } x \in X \text{ and } x \neq x^*, y \in Y \end{cases}$$

We have $\mu_{A\circ R}(y) = \max_{x\in X} \min\ [\mu_A(x)\ ,\ \mu_R(x\ ,y\ )]$

$= \min[\mu_A(x^*), \mu_R(x^*,y)] = \min[\mu_A(x^*), \mu_B(y)] = \mu_B(y)$.

~ α – operation :

For any a , b $\in$ [0 , 1], the α - operator is defined as

$$a \ \alpha \ b \triangleq \begin{cases} 1 & \text{if } a \le b \\ b & \text{if } a > b . \end{cases}$$

• *α - Composition :*

~ For fuzzy sets A and B in X and Y, respectively , the α - composition of A and B forms a fuzzy relation $A \overset{\alpha}{\leftrightarrow} B$ in X$\times$Y which is defined by :

$$\mu_{A \overset{\alpha}{\longleftrightarrow} B}(x,y) \triangleq \mu_A(x) \, \alpha \, \mu_B(y) = \begin{cases} 1 & \text{if } \mu_A(x) \le \mu_B(y) \\ \mu_B(y) & \mu_A(x) > \mu_B(y) \end{cases}$$

~ The α – composition of a fuzzy relation R and a fuzzy set B is denoted by $R \overset{\alpha}{\leftrightarrow} B$, and is defined by

$$\mu_{R \overset{\alpha}{\longleftrightarrow} B}(x) \triangleq \min_{y \in Y}[\mu_R(x,y) \ \alpha \ \mu_B(y)]$$

Thm 3.2:

Let R be a fuzzy relation on X$\times$Y. For any fuzzy sets A and B in X and Y , respectively, we have

$$R \subseteq A \overset{\alpha}{\leftrightarrow} (A \circ R) \ \cdots\cdots\cdots\cdots\cdot (*)$$

$$A \circ (A \overset{\alpha}{\leftrightarrow} B) \subseteq B \ \cdots\cdots\cdots\cdots\cdots\cdot (**)$$

Pf: (*) :

$$\mu_{A \overset{\alpha}{\longleftrightarrow} (A \circ R)}(x, y) = \mu_A(x) \ \alpha \ \mu_{A \circ R}(y) = \mu_A(x) \ \alpha \ [\max_{z \in X}(\mu_A(z) \wedge \mu_R(z, y))]$$

$$= \mu_A(x) \ \alpha \ \{ \max [\max_{z \in X, z \ne x} (\mu_A(z) \wedge \mu_R(z,y)), \mu_A(x) \wedge \mu_R(x,y)]\}$$

$$\ge \mu_A(x) \ \alpha \ [\mu_A(x) \wedge \mu_R(x,y)] \ge \mu_R(x,y) .$$

(**) :

$$\mu_{A\circ(A\overset{\alpha}{\longleftrightarrow}B)}(y) = \max_{x\in X}[\,\mu_A(x) \wedge \mu_{A\overset{\alpha}{\longleftrightarrow}B}(x,y)]$$

$$= \max_{x\in X}\{\mu_A(x) \wedge [\,\mu_A(x)\ \alpha\ \mu_B(y)]\,\} \le \mu_B(y)\ .$$

Thm 3.3 :

If the solution of P2 exists then the largest R (in the sense of set – theoretic inclusion) that satisfies the fuzzy relation equation $A\circ R = B$ is $\hat{R}$.

$$\hat{R} = A \overset{\alpha}{\leftrightarrow} B\ .$$

Pf : from (**), $A\circ(A\overset{\alpha}{\leftrightarrow}B)\subseteq B$, we have $A\circ\hat{R} \subseteq B$ ······· (1)

from (*), $R\subseteq A\overset{\alpha}{\leftrightarrow}(A\circ R)$. If $A\circ R = B$, we have $R \subseteq \hat{R}$ ······· (2)

By the monotonicity of max–min composition. We obtain

$$A\circ R \subseteq A\circ\hat{R},\ i.e.\ \ B\subseteq A\circ\hat{R}\ \cdots\cdots\cdots(3)$$

from (1) and (3) , $A\circ\hat{R} = B$.

from (2) , $\hat{R}$ is the largest solution to $A\circ R = B$.

Ex 3.22:

From Ex 3.21, $A = 0.2/x_1 + 0.8/x_2 + 1/x_3$, $B = 0.5/y_1 + 0.8/y_2 + 0.6/y_3$.

$$\hat{R}=A\overset{\alpha}{\leftrightarrow}B = \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\begin{pmatrix} 0.2 \\ 0.8 \\ 1 \end{pmatrix} \overset{\alpha}{\leftrightarrow} \begin{array}{ccc} y_1 & y_2 & y_3 \\ \end{array}\!\begin{pmatrix} 0.5 & 0.8 & 0.6 \end{pmatrix} = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 1 & 1 & 1 \\ x_2 & 0.5 & 1 & 0.6 \\ x_3 & 0.5 & 0.8 & 0.6 \end{array}$$

check : $R\subset\hat{R}$

Thm 3.4 :

Problem P3 has no solution if :

$$\max_{x\in X}\mu_R(x,\ y) < \mu_B(y)\ .$$

Remark : $\mu_B(y) = \max_{x\in X}\min\,[\,\mu_A(x),\ \mu_R(x\ ,\ y)\,]$.

Ex 3.23 :

$$A \circ R = (\mu_A(x_1)\ \mu_A(x_2)\ \mu_A(x_3)) \circ \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 0.8 & 0.5 \\ x_2 & 1 & 0.7 \\ x_3 & 0.3 & 0.2 \end{array} = (0.5 \quad 1) = B \ .$$

$$\max_{x \in X} \mu_R(x, y_2) = \max(0.5 \quad 0.7 \quad 0.2) = 0.7 < \mu_B(y_2) = 1$$

$\therefore$ no solution

Thm 3.6 :

If a solution to problem P3 exists, then the largest fuzzy set A that satisfies $A \circ R = B$ is $\hat{A}$:

$$\hat{A} = R \overset{\alpha}{\leftrightarrow} B$$

whose membership function is given by

$$\mu_{R \overset{\alpha}{\longleftrightarrow} B}(x) \triangleq \min_{y \in Y} \ [\ \mu_{\mathrm{R}}(\mathrm{x}\ ,\ \mathrm{y})\ \alpha\ \mu_{\mathrm{B}}(\mathrm{y})\ ]\ .$$

Ex3.24 :

$$\hat{A} = R \overset{\alpha}{\leftrightarrow} B(x) \triangleq \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.7 & 1 & 0.4 \\ x_2 & 0.5 & 0.9 & 0.6 \\ x_3 & 0.2 & 0.6 & 0.3 \end{array} \overset{\alpha}{\leftrightarrow} \begin{array}{c|c} y_1 & 0.5 \\ y_2 & 0.8 \\ y_3 & 0.6 \end{array}$$

$$= \begin{pmatrix} 0.5 \wedge 0.8 \wedge 1 \\ 1 \wedge 0.8 \wedge 1 \\ 1 \wedge 1 \wedge 1 \end{pmatrix} = \begin{array}{c|c} x_1 & 0.5 \\ x_2 & 0.8 \\ x_3 & 1 \end{array}$$