Chapter 1. Basics of Fuzzy Sets.

- ***Set:***

A classical (crisp) set can be defined (within a universe of discourse or universal set U) by :

1. naming all its members (used only for finite sets).

   e.g. A={a1, a2, ……., an}

2. a property satisfied by its members,

   A={x| p(x)}.   p(x): property of x.

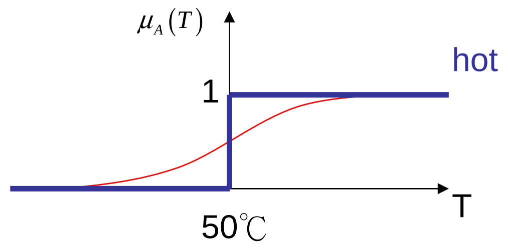
3. by a characteristic function, $\mu_A(x)$, as follows

$$\mu_A(x) = \begin{cases} 1 & iff \quad x \in A \\ 0 & iff \quad x \notin A \end{cases}$$

$$i.e. \quad \mu_A : U \rightarrow \{0,1\}$$

- ***Fuzzy set :***

~ A fuzzy set introduces vagueness by eliminating the sharp boundary that divides members from nonmembers in the group, i.e. the transition between membership and nonmembership is gradual rather than abrupt.



~ A fuzzy set in U can be defined as set of ordered pairs:

$$\widetilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in U\} \qquad \mu_{\tilde{A}} : U \rightarrow M.$$

where $\mu_{\tilde{A}}(\bullet)$: membership function (or characteristic function) of Ã .
$\mu_{\tilde{A}}(x)$ is the grade (or degree) that x belong to Ã.
M: membership space.

For crisp set : M = {0,1}
fuzzy set: M is a subset of nonnegative real numbers whose supremum is finite. In general M:[0,1].

Ex2-1:

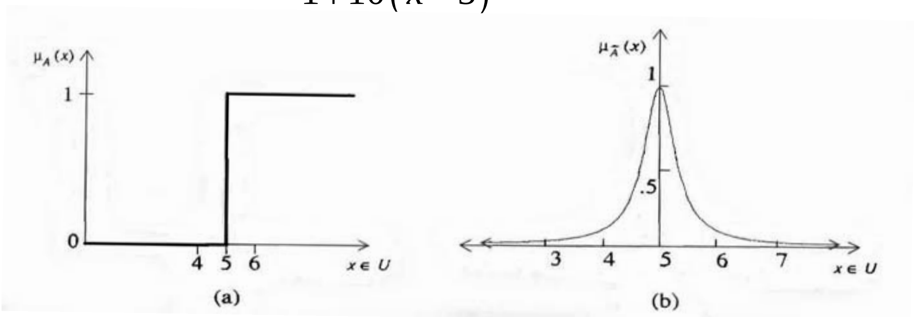
1. U:**R,** crisp set A represents " real numbers ≧ 5"

$$A = \{(x, \mu_A(x)) \mid x \in U\}$$

$$and \qquad \mu_A(x) = \begin{cases} 0 & , x < 5 \\ 1 & , x \geq 5 \end{cases}$$

2. U:**R,** fuzzy set $\tilde{A}$ represents " real numbers close to 5."
   We have $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in U\}$

$$\mu_{\tilde{A}}(x) = \frac{1}{1+10(x-5)^2}$$



Remark:

① the assignment of the membership function of a fuzzy set is subjective. ∴ in the above example, we may assign :

$$\mu_{\tilde{A}}(x) = \frac{1}{1+(x-5)^2}$$

② $\mu_{\tilde{A}}(x)$ is not a probability because $\Sigma \mu_{\tilde{A}}(x) \neq 1$.

• *Support of a fuzzy set:*

A crisp set s.t. $Supp(A) = \{x \in U \mid \mu_A(x) > 0\}$

Ex 2.2:

① U:{10, 20,…,100}. Fuzzy sets A≡ "High Score,"
B≡ "Medium Score," C≡ "Low Score."
Membership function :

Numerical Score	A	B	C
10	0	0	1
20	0	0	1
30	0	0.1	0.9
40	0	0.5	0.7
50	0.1	0.8	0.5
60	0.3	1	0.3
70	0.5	0.8	0.1
80	0.8	0.5	0
90	1	0	0
100	1	0	0

Then:

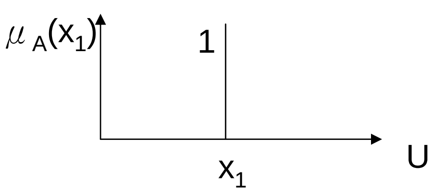
$$Supp(A)=\{50,60,70,80,90,100\}$$
$$Supp(B)=\{30,40,50,60,70,80\}$$
$$Supp(C)=\{10,20,30,40,50,60,70\}$$

② U=[0,100]

$$\mu_A(x)=\mu_{High-Score}(x)=\begin{cases}0 & 0\le x\le 40\\ \left\{1+\left[\left(x-40\right)/5\right]^{-2}\right\}^{-1} & 40< x\le 100\end{cases}$$

$$Supp(A)=(40,100]$$

- ***Fuzzy singleton:***
  Supp(A) is a single point x_1 in U with $\mu_A(x_1)=1$.



- ***Crossover point:***

$$\{x \in U \mid \mu_A(x) = 0.5\}$$

- ***Kernel:***

$$Ker(A) = \{x \in U \mid \mu_A(x) = 1\}$$

- ***Height:***
  Height of A= Height (A) $\equiv \sup_x \mu_A(x)$.

- ***Normal and Subnormal:***
  A fuzzy set A is normal if Height (A)=1.
  Otherwise it is subnormal.

- Representation of a fuzzy set:

~ For a discrete U={x_1, x_2,…, x_n}, a fuzzy set A can be written as

① A={(x_1, $\mu_A(x_1)$), (x_2, $\mu_A(x_2)$) ,…, (x_n, $\mu_A(x_n)$)}.
e.g. :
B= "Medium Score" in Ex 2.2.
={(10, 0), (20, 0), (30, 0.1), (40, 0.5), (50, 0.8), (60, 1), (70, 0.8), (80, 0.5), (90, 0), (100, 0)}

② $A = \mu_1/x_1 + \mu_2/x_2 + \cdots + \mu_n/x_n = \sum_{i=1}^{n} \mu_i/x_i$

where $\mu_i = \mu_A(x_i) > 0$

e.g. : B=0.1/30+0.5/40+0.8/50+1/60+0.8/70+0.5/80.

~ if U is **R,** then A is written as

$$A = \int_U \mu_A(x)/x$$

e.g. : A in Ex 2.1

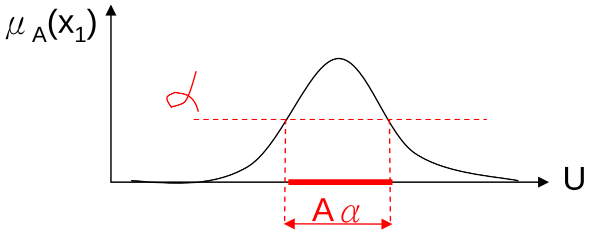
$$A = \int_R \frac{1}{1+10(x-5)^2} \Big/ x$$

- ***α*-cut (or *α*-level set) of A:**

A crisp set A_α s. t.

$$A_\alpha = \{x \in U \mid \mu_A(x) \geq \alpha\}, \qquad \alpha \in (0,1].$$

if $A_\alpha = \{x \in U \mid \mu_A(x) > \alpha\}$, A_α is called a strong α-cut



- ***level set of A:***

the set of all levels $\alpha \in (0,1]$ that represents distinct α-cut of a given fuzzy set A,

i.e.

$$\Lambda_A = \{\alpha \mid \mu_A(x) = \alpha, \quad for \quad some \quad x \in U\}.$$

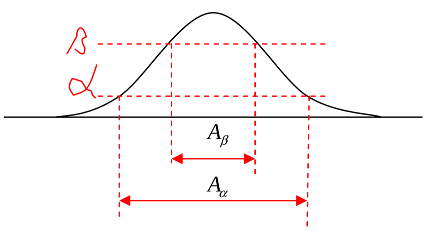
Ex 2.3:

In Ex 2.2.

$$A_{0.5} = High - Score_{0.5} = \{70, 80, 90, 100\}.$$

$$\Lambda_A = \{0.1, 0.3, 0.5, 0.8, 1\}$$

Remark: if $\alpha \leq \beta$, *then* $A_\beta \subseteq A_\alpha$



Thm 2.1:

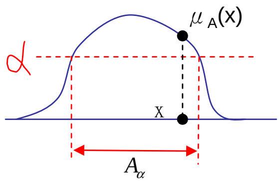
Let A be a fuzzy set in U. Then $\mu_A(x)$ can be expressed as

$$\mu_A(x) = \sup_{\alpha \in (0,1]} \left[\alpha \wedge \mu_{A_\alpha}(x)\right]. \qquad \forall x \in U$$

where $\wedge$ denotes the min operation and

$$\mu_{A_\alpha}(x)=\begin{cases}1 & iff \quad x\in A_\alpha \\ 0 & otherwise\end{cases}$$

pf: let $\vee$ denote the max operation.



$$\begin{aligned}
&\sup_{\alpha\in(0,1]}\left[\alpha\wedge\mu_{A_\alpha}(x)\right] \\
&=\sup_{\alpha\in(0,\mu_A(x)]}\left[\alpha\wedge\mu_{A_\alpha}(x)\right]\vee\sup_{\alpha\in(\mu_A(x),1]}\left[\alpha\wedge\mu_{A_\alpha}(x)\right] \\
&=\sup_{\alpha\in(0,\mu_A(x)]}[\alpha\wedge 1]\vee\sup_{\alpha\in(\mu_A(x),1]}[\alpha\wedge 0] \\
&=\sup_{\alpha\in(0,\mu_A(x)]}\alpha \\
&=\mu_A(x).
\end{aligned}$$

- ***Resolution principle*** *(decomposition thm. or representation thm.):*
  Let A be a fuzzy set and denote a fuzzy set with

$$\mu_{\alpha A_\alpha}(x)=\alpha\wedge\mu_{A_\alpha}(x), \qquad \forall x\in U$$

Then the resolution principle states A can be expressed by

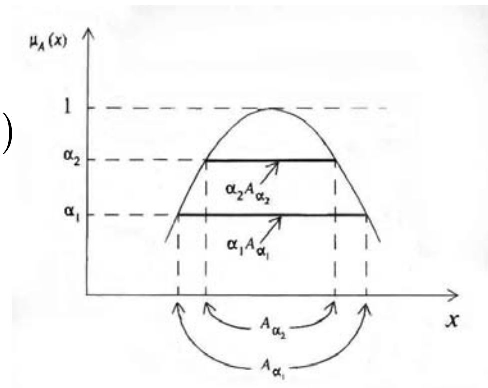
$$A=\bigcup_{\alpha\in\Lambda_A}\alpha A_\alpha \qquad or \qquad A=\int_0^1\alpha A_\alpha$$

where U: union.

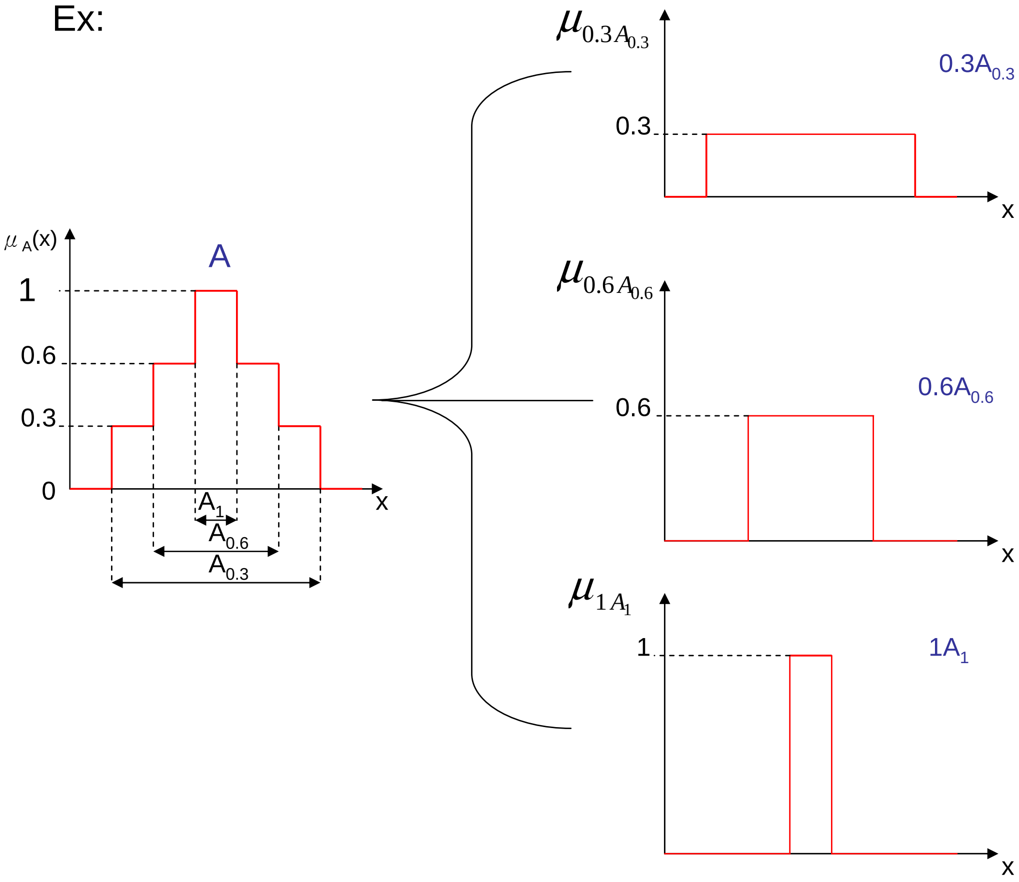
Pf: by Thm 2.1

$$\sup_{\alpha\in\Lambda_A}\left[\alpha\wedge\mu_{A_\alpha}(x)\right]=\sup_{\alpha\in\Lambda_A}\mu_{\alpha A_\alpha}(x)=\mu_A(x)$$

$$\Rightarrow\bigcup_{\alpha\in\Lambda_A}\alpha A_\alpha=A$$



Ex:



Ex 2.4:

In Ex 2.2. we have A=0.1/50+0.3/60+0.5/70+0.8/80+1/90+1/100.

$$
\begin{aligned}
A &= 0.1/50 + 0.1/60 + 0.1/70 + 0.1/80 + 0.1/90 + 0.1/100 \\
&\quad + 0.3/60 + 0.3/70 + 0.3/80 + 0.3/90 + 0.3/100 \\
&\quad + 0.5/70 + 0.5/80 + 0.5/90 + 0.5/100 \\
&\quad + 0.8/80 + 0.8/90 + 0.8/100 \\
&\quad + 1/90 + 1/100 \\
&= 0.1\left(1/50 + 1/60 + 1/70 + 1/80 + 1/90 + 1/100\right) \\
&\quad + 0.3\left(1/60 + 1/70 + 1/80 + 1/90 + 1/100\right) \\
&\quad + 0.5\left(1/70 + 1/80 + 1/90 + 1/100\right) \\
&\quad + 0.8\left(1/80 + 1/90 + 1/100\right) \\
&\quad + 1\left(1/90 + 1/100\right) \\
&= 0.1A_{0.1} + 0.3A_{0.3} + 0.5A_{0.5} + 0.8A_{0.8} + 1A_1 \\
&= \bigcup_{\alpha \in \Lambda_A} \alpha A_\alpha \quad \text{(resolution principle)}
\end{aligned}
$$

$$\Lambda_A = \{0.1, 0.3, 0.5, 0.8, 1\}$$

On the other hand, given

$$A_{0.1}=\{1,2,3,4,5\}.\quad A_{0.4}=\{2,3,5\}.\quad A_{0.8}=\{2,3\}.\quad A_{1}=\{3\}.$$

By representation thm.:

$$A=\bigcup_{\alpha\in\Lambda_A}\alpha A_\alpha=\bigcup_{\alpha\in\{0.1,0.4,0.8,1\}}\alpha A_\alpha$$

$$=0.1A_{0.1}+0.4A_{0.4}+0.8A_{0.8}+1A_1$$

$$=0.1\left(1/1+1/2+1/3+1/4+1/5\right)$$

$$+0.4\left(1/2+1/3+1/5\right)$$

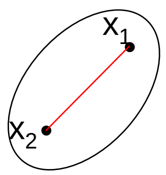
$$+0.8\left(1/2+1/3\right)$$

$$+1\left(1/3\right)$$

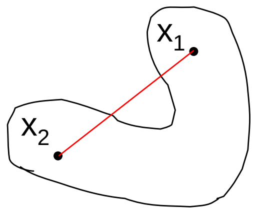
$$=0.1/1+0.8/2+1/3+0.1/4+0.4/5.$$

- ***Convex set:***

  A set S is convex if $\forall x_1,x_2\in S.\quad \alpha\in\mathbb{R}\quad and\quad 0<\alpha<1.$
  the point $\alpha x_1+(1-\alpha)x_2\in S$
  e.g. :



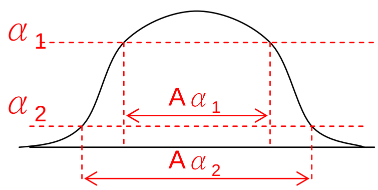
convex



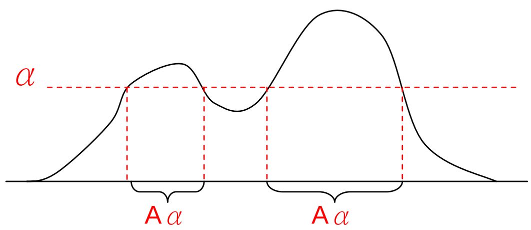
nonconvex

- ***Convex fuzzy set:***

  if the α-cuts of a fuzzy set are convex for all $\alpha\in(0,1]$ then the fuzzy set is convex.
  e.g. :



convex fuzzy set.

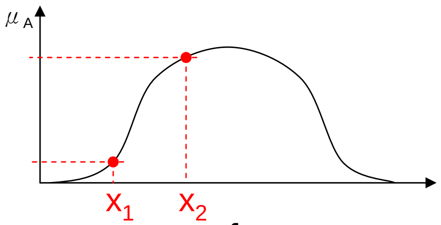


nonconvex fuzzy set.

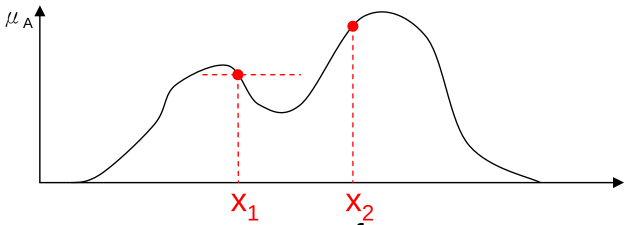
Thm: A fuzzy set A on **R** is convex iff

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min[\mu_A(x_1), \mu_A(x_2)]$$

$$for \quad all \quad x_1, x_2 \in \mathbb{R}, \quad \lambda \in [0,1].$$



convex fuzzy set



nonconvex fuzzy set

Pf:

$(1) \Rightarrow$

$$let \quad \alpha = \mu_A(x_1) \leq \mu_A(x_2). \quad then \quad x_1, x_2 \in A_\alpha.$$

$$by \quad the \quad convexity \quad of \quad A. \quad \lambda x_1 + (1-\lambda)x_2 \in A_\alpha. \quad \forall \lambda \in [0,1]$$

$$\therefore \quad \mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \alpha = \mu_A(x_1) = \min[\mu_A(x_1), \mu_A(x_2)]$$

$(2) \Leftarrow$

$$Here, \quad we \quad need \quad to \quad prove \quad that \quad \forall \alpha \in (0,1], \quad A_\alpha \quad is \quad convex.$$

$$\forall x_1, x_2 \in A_\alpha \quad \left(i.e. \quad \mu_A(x_1) \geq \alpha, \quad \mu_A(x_2) \geq \alpha \right). \quad \forall \lambda \in [0,1].$$

$$we \quad have \quad \mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min[\mu_A(x_1), \mu_A(x_2)] \geq \min(\alpha, \alpha) = \alpha$$

$$\Rightarrow \quad \lambda x_1 + (1-\lambda)x_2 \in A_\alpha$$

$$\therefore \quad A_\alpha \quad is \quad convex \quad \forall \alpha \in (0,1]$$

$$\therefore \quad A \quad is \quad convex$$

Remark:

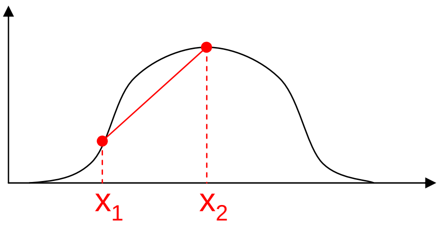
(1) Convex function:

A function f defined on a convex set S is said to be convex if, $\forall x_1, x_2 \in S, \quad \lambda \in [0,1],$ there holds

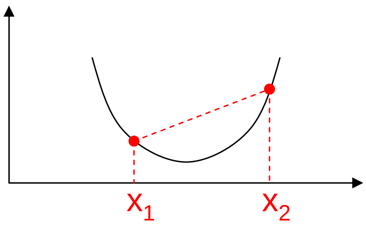
$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2).$$

(2) The convexity of a fuzzy set does not mean that the membership function of a convex fuzzy set is a convex function.

e.g. :



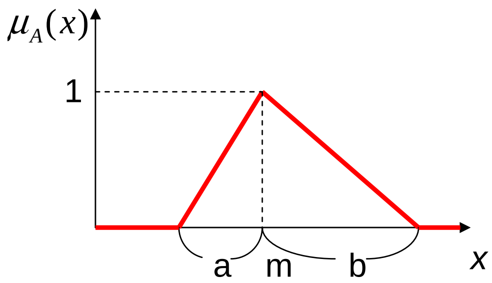
Convex fuzzy set,
nonconvex function



nonconvex fuzzy set,
convex function

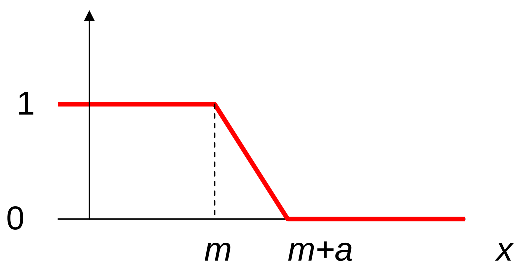
- ***Fuzzy number:***
  A convex and normal fuzzy set defined on **R** whose membership function is piecewise continuous or, equivalently, each α-cut is a closed interval, is called a fuzzy number.
  e.g. Triangular membership function.

$$\mu_A(x;m,a,b)=\begin{cases}1+\dfrac{x-m}{a}, & m-a\le x\le m\\ 1+\dfrac{m-x}{b}, & m\le x\le m+b\\ 0, & otherwise\end{cases}$$



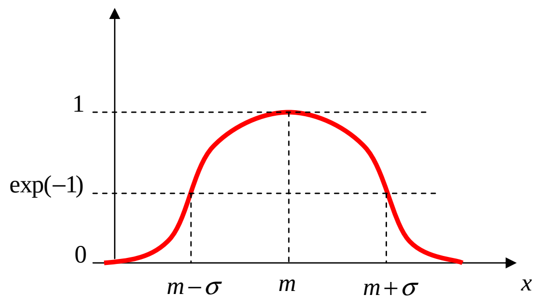
~Trapezoid membership function

$$\mu_A(x;a,m)=\begin{cases}0, & m+a\le x\\ 1+\dfrac{m-x}{a}, & m\le x\le x+a\\ 1, & otherwise\end{cases}$$



~ Gaussian membership function

$$\mu_A(x;m,\sigma)=\exp\{-(\frac{x-m}{\sigma})^2\}$$



- ***Cardinality:***

~ crisp set: the number of elements in the crisp set.

~ fuzzy set: $|A| = \sum_{x \in U} \mu_A(x).$

~ The relative cardinality of A is

$$|A|_{rel} = \frac{|A|}{|U|}$$, where $|U|$ is finite.

~ When a fuzzy set A has a finite support, its cardinality can be defined as a fuzzy set. This fuzzy cardinality is defined as:

$$|A|_f = \sum_{\alpha \in \Lambda_A} \alpha / |A_\alpha|$$

Ex 2.5:
Consider the fuzzy set in Ex 2.2 (p 1-5).

$$|A| = |High-score| = 0.1+0.3+0.5+0.8+1+1 = 3.7$$

$$|A|_{ref} = \frac{|A|}{|U|} = \frac{|A|}{10} = 0.37$$

$$|A|_f = 0.1/6+0.3/5+0.5/4+0.8/3+1/2$$

- ***Set-theoretic definitions and operations for fuzzy set***

1. *Complement:*

$$\mu_A(x) \in [0,1].$$ $\bar{A}$: complement of A

$$\mu_{\bar{A}}(x) \triangleq 1 - \mu_A(x). \qquad \forall x \in U$$

2. *Intersection:*
denoted as A∩B.

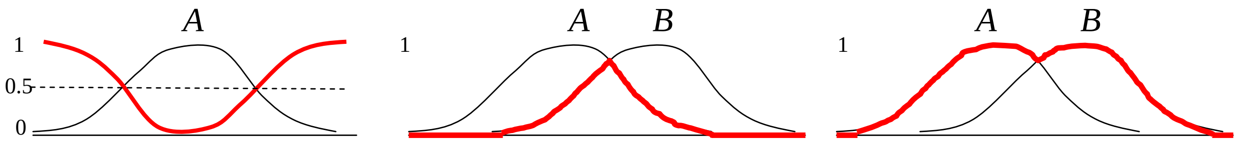
$$\mu_{A \cap B}(x) \triangleq \min\left[\mu_A(x), \mu_B(x)\right] \equiv \mu_A(x) \wedge \mu_B(x) \qquad \forall x \in U$$

Remark : $A \cap B \subseteq A$ *and* $A \cap B \subseteq B$

3. *Union:*

$$A \cup B, \quad \mu_{A \cup B}(x) \triangleq \max\left[\mu_A(x), \mu_B(x)\right] \equiv \mu_A(x) \vee \mu_B(x) \quad \forall x \in U$$

Remark : $A \subseteq A \cup B$ *and* $B \subseteq A \cup B$



$(a)\ \overline{A}.$ $(b)\ A\cap B.$ $(c)\ A\cup B.$

4. *Equality:*
   A and B are equal iff $\mu_A(x)=\mu_B(x),\quad \forall x\in U$

   Remark: To check the degree of equality of two fuzzy sets, we can use the *similarity measure:*

$$E(A,B)\equiv \deg ree(A=B)\triangleq \frac{|A\cap B|}{|A\cup B|}$$

$$when\quad A=B,\quad E(A,B)=1.$$

$$when\quad |A\cap B|=0.\quad \left(A\quad and\quad B\quad do\quad not\quad overlap\right)$$

$$E(A,B)=0.$$

5. *Subset:*
   A is a subset of B, i.e. $A\subseteq B\quad iff$

$$\mu_A(x)\le \mu_B(x),\quad \forall x\in U$$

   A is a proper subset of B if $A\subset B.$
   Remark: subsethood measure:

$$s(A,B)=\deg ree(A\subseteq B)\triangleq \frac{|A\cap B|}{|A|}$$

   Remark: With the above definitions, we have

$$(A\cap B)_\alpha = A_\alpha\cap B_\alpha,\quad (A\cup B)_\alpha = A_\alpha\cup B_\alpha,$$

$$\text{but}\quad (\overline{A})_\alpha \ne \overline{A_\alpha}$$

6. *Double-negation law (involution):*

$$\overline{\overline{A}}=A.$$

7. *De Morgan's laws:*

$$\overline{A \cup B} = \overline{A} \cap \overline{B}, \quad \overline{A \cap B} = \overline{A} \cup \overline{B}.$$

Remark:

1. the law of the excluded middle $\left(i.e.\ E \cup \overline{E} = U\right)$ and the law of contradication $\left(i.e.\ E \cap \overline{E} = \varnothing\right)$ are no longer true in fuzzy sets. That is,

$$A \cup \overline{A} \neq U, \quad A \cap \overline{A} \neq \varnothing.$$

2. The above definitions are not unique, we will see other definitions later in this chapter.

8. *Cartesian product:*

Let A_1, A_2, … , A_n be fuzzy sets in U_1, U_2, …, U_n, respectively. The Cartesian product of A_1, A_2, … , A_n is a fuzzy set in the product space $U_1 \times U_2 \times \cdots \times U_n$ with

$$\mu_{A_1 \times A_2 \times \cdots \times A_n}(x_1, x_2, \cdots, x_n) \triangleq \min\left[\mu_{A_1}(x_1), \mu_{A_2}(x_2), \cdots, \mu_{A_n}(x_n)\right]$$

$$x_1 \in U_1,\ x_2 \in U_2 \cdots, x_n \in U_n.$$

9. *Algebraic sum:*

$$A + B. \qquad \mu_{A+B}(x) \triangleq \mu_A(x) + \mu_B(x) - \mu_A(x) \bullet \mu_B(x).$$

10. *Algebraic product:*

$$A \bullet B. \qquad \mu_{A \bullet B}(x) \triangleq \mu_A(x) \bullet \mu_B(x).$$

11. *Bounded sum:*

$$A \oplus B. \qquad \mu_{A \oplus B}(x) \triangleq \min\left[1, \mu_A(x) + \mu_B(x)\right].$$

12. *Bounded difference:*

$$A \ominus B \qquad \mu_{A \ominus B}(x) \triangleq \max\left[0, \mu_A(x) - \mu_B(x)\right].$$

Ex2.7:

$$A = 0.5/3 + 1/5 + 0.6/7, \quad B = 1/3 + 0.6/5$$

$$A \times B = 0.5/(3,3) + 1/(5,3) + 0.6/(7,3) + 0.5/(3,5) + 0.6/(5,5) + 0.6/(7,5).$$

$$A + B = 1/3 + 1/5 + 0.6/7$$

$$A \bullet B = 0.5/3 + 0.6/5$$

$$A \oplus B = 1/3 + 1/5 + 0.6/7$$

$A \ominus B = 0.4/5 + 0.6/7.$

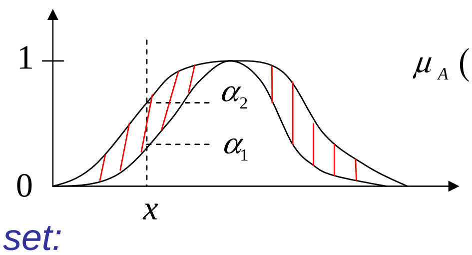
- ***Other types of fuzzy sets:***

~ *interval-valued fuzzy sets:*

μ_A (x) is not a real number but a closed interval of real numbers between the identified lower and upper bounds, i.e.

$$\mu_A : \quad U \rightarrow \varepsilon([0,1])$$

where $\varepsilon([0,1])$ denotes the family of all closed intervals in [0,1].



$$\mu_A(x) = [\alpha_1, \ \alpha_2]$$

- *Type 2 fuzzy set:*

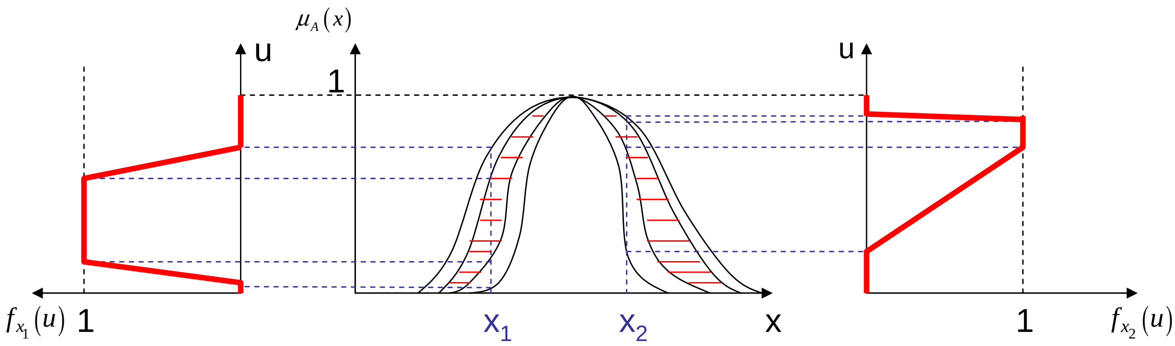
Interval-valued fuzzy sets can further be generalized by allowing their intervals to be fuzzy. A type-2 fuzzy set is a fuzzy set whose membership values are (type-1) fuzzy sets on [0,1], i.e.

$$\mu_A : \ U \rightarrow [0,1]^{[0,1]}$$

where μ_A (x) is a fuzzy set in [0,1] represented by

$$\mu_A(x) = \int f(u)/u, \qquad u \in [0,1].$$

$$f : [0,1] \ \rightarrow \ [0,1].$$



~ Remark: we can recursively define a type-m fuzzy set (m>1) in U whose membership values are type m-1 fuzzy sets on [0, 1].

Ref: Jerry M. Mendel, " Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions," Prentice-Hall, 2001.

• *Level-2 fuzzy set*:
A fuzzy set whose elements are fuzzy sets.

$$\mu_A : \tilde{p}(U) \rightarrow [0,1]$$

where $\tilde{p}(U)$: the set of all fuzzy subsets of U.

• *Further operations on fuzzy sets:*

~ Different functions are defined for fuzzy sets operation and can be divided into two categories. One is non-parametric function and the other is parametric function in which parameters are used to adjust the " strength" of the corresponding operations.

~ *Complement:*

$\bar{A}$ is specified by a function

$$c : [0,1] \rightarrow [0,1]$$

s.t. $\mu_{\bar{A}}(x) = c(\mu_A(x))$.

where c(•) satisfies:
c1: Boundary conditions : c(0)=1, c(1)=0

c2: Monotonic property:
$\forall$ x_1 , $x_2 \in$ U. if $\mu_A(x_1) \le \mu_A(x_2)$, then
$c(\mu_A(x_1)) \ge c(\mu_A(x_2))$.
C3: Continuity: c(•) is a continuous function.
C4: Involution: $c(c(\mu_A(x))) = \mu_A(x)$, $\forall$ x $\in$ U

Based on the above conditions, typical functions are:

1. Negation complement:

$$\bar{A} : \mu_{\bar{A}}(x) = c(\mu_A(x)) \triangleq 1 - \mu_A(x), \ \forall x \in \mathrm{U}$$

2. λ Complement (Sugeno class):

$$\bar{A}^{\lambda} \ : \ \mu_{\bar{A}^{\lambda}}(x) = c(\mu_A(x)) \triangleq \frac{1-\mu_A(x)}{1+\lambda\mu_A(x)}. \quad -1 < \lambda < \infty.$$

Observation: λ =0, $c(\mu_A(x)) = 1 - \mu_A(x)$, $\lambda \rightarrow -1$, $\bar{A}^{\lambda} \rightarrow$ U.
$\lambda \rightarrow \infty$, $\bar{A}^{\lambda} \rightarrow \varnothing$.

3. w Complement (Yager class):

$$\bar{A}^{w} : \mu_{\bar{A}^{w}}(x) = c(\mu_A(x)) \triangleq (1 - \mu_A^{w}(x))^{\frac{1}{w}}, \quad 0 < w < \infty$$

observation : w=1, $c(\mu_A(x)) = 1 - \mu_A(x)$.

Remark: The *equilibrium* of a fuzzy complement c is defined as any value for which c(a)=a, e.g. a=0.5 in standard complement operation. Every fuzzy complement has at most one equilibrium due to the monotonic property.

~ *Intersection:*

often referred to as triangular norms (t- norms):

t- norms are of the form:

$$t : [0,1] \times [0,1] \rightarrow [0,1]$$

s.t. $\mu_{A \cap B}(x) = t[\mu_A(x), \mu_B(x)]$

where t(• , •) satisfies:

$t1$. Boundary conditions:

$$t(0,0)=0, \quad t(\mu_A(x),1)=t(1,\mu_A(x))=\mu_A(x).$$

$t2$. Commutativity:

$$t(\mu_A(x),\mu_B(x))=t(\mu_B(x),\mu_A(x)).$$

$t3$. Monotonicity:

if $\mu_A(x) \le \mu_C(x)$ and $\mu_B(x) \le \mu_D(x)$, then

$$t(\mu_A(x),\mu_B(x)) \le t(\mu_C(x),\mu_D(x)).$$

$t4$. Associativity:

$$t(\mu_A(x), t(\mu_B(x),\mu_C(x))) = t(t(\mu_A(x),\mu_B(x)),\mu_C(x)).$$

Let a = $\mu_A(x)$, b= $\mu_B(x)$, typical non-parametric t-norms are:

1. Intersection: a∧b = min(a , b).
2. Algebraic product: a • b = ab
3. Bounded product: a⊙b = max(0 , a+b-1)
4. Drastic product: $a \hat{\cdot} b = \begin{cases} a & b=1 \\ b & a=1 \\ 0 & a,b<1 \end{cases}$

~ typical parametric t-norm:

5. Yager intersection:

$$t_w(a,b) = 1 - \min\left[1,\ \left((1-a)^w + (1-b)^w\right)^{\frac{1}{w}}\right] \qquad w \in (0,\infty)$$

Observation: w = 1 : bounded product.

$w \to \infty : t_w(a,b) = \min(a,b)$

$w \to 0$: drastic product.

~ *Union:*

often referred to as triangular conorms (t- conorms).

t- conorms are of the form.

S: [0,1]×[0,1] → [0,1].

s.t. $\mu_{A \cup B}(x) = s[\mu_A(x), \mu_B(x)]$.

where $s(\bullet, \bullet)$ satisfies:

s1. Boundary conditions:

$s(1,1) = 1, \quad s(\mu_A(x), 0) = s(0, \mu_A(x)) = \mu_A(x)$.

s2. Commutativity: $s(\mu_A(x), \mu_B(x)) = s(\mu_B(x), \mu_A(x))$.

s3. Monotonicity:

if $\mu_A(x) \le \mu_C(x), \quad \mu_B(x) \le \mu_D(x)$, then

$s(\mu_A(x), \mu_B(x)) \le s(\mu_C(x), \mu_D(x))$.

s4. Associativity:

$s(\mu_A(x), s(\mu_B(x), \mu_C(x))) = s(s(\mu_A(x), \mu_B(x)), \mu_C(x))$.

~ typical nonparametric t- conorms are

1. Union: $a \vee b = \max(a,b)$
2. Algebraic sum: $a \hat{+} b = a + b - ab$
3. Bounded sum: $a \oplus b = \min(1, a+b)$
4. Drastic sum: $a \dot{\vee} b = \begin{cases} a & , b = 0 \\ b & , a = 0 \\ 1 & , a, b > 0 \end{cases}$
5. Disjoint sum: $a \Delta b = \max\{\min(a, 1-b), \min(1-a, b)\}$.

~ typical parametric t- conorm:

6. Yager union : $S_w(a,b) = \min[1, (a^w + b^w)^{1/w}]$. $w \in (0, \infty)$

 $w = 1, \quad S_w(a,b) = a \oplus b$.

 $w \to 0, \quad S_w(a,b) = a \dot{\vee} b$

 $w \to \infty, \quad S_w(a,b) = a \vee b$

Thm 2.2. :

$a \dot{\wedge} b = t_{min}(a, b) \le t(a, b) \le t_{max}(a, b) = \min(a, b)$ -------- (1)

$\max(a, b) = S_{min}(a, b) \le S(a, b) \le S_{max}(a, b) = a \dot{\vee} b$

pf of (1) :

2nd inequality:

By boundary condition, $t(a, 1) = a$, $t(1, b) = b$.

By monotonicity condition, $t(a, b) \le t(a, 1) = a$.

$t(a, b) \le t(1, b) = b$

$\Rightarrow t(a, b) \le \min(a, b)$

1st inequality: when $b = 1$, $t(a, b) = a = a \hat{\wedge} b$

$a = 1$, $t(a, b) = b = a \hat{\wedge} b$

$\therefore$ hold for $a = 1$ or $b = 1$.

When $a, b < 1$, $t(a, b) \ge 0 = a \hat{\wedge} b$

$\Rightarrow$ $t(a, b) \ge a \hat{\wedge} b$.

Remark: the Yager intersection and Yager union become

t_{max} and S_{min} as $w \to \infty$ and become

t_{min} and S_{max} as $w \to 0$.

- ***Aggregation operation:***

defined by $h: [0,1]^n \to [0,1]$, $n \ge 2$.

s.t. $\mu_A(x) = h(\mu_{A_1}(x), \mu_{A_2}(x), \cdots\cdots, \mu_{A_n}(x))$. $\forall x \in U$.

~ t- norms and t- conorms are a kind of aggregation operations on fuzzy sets, which are operations by which several fuzzy sets are combined to produce a single set.

~ *averaging operators:*

aggregation operations for which

$$\min(a_1, a_2, \cdots, a_n) \le h(a_1, a_2, \cdots, a_n) \le \max(a_1, a_2, \cdots, a_n)$$

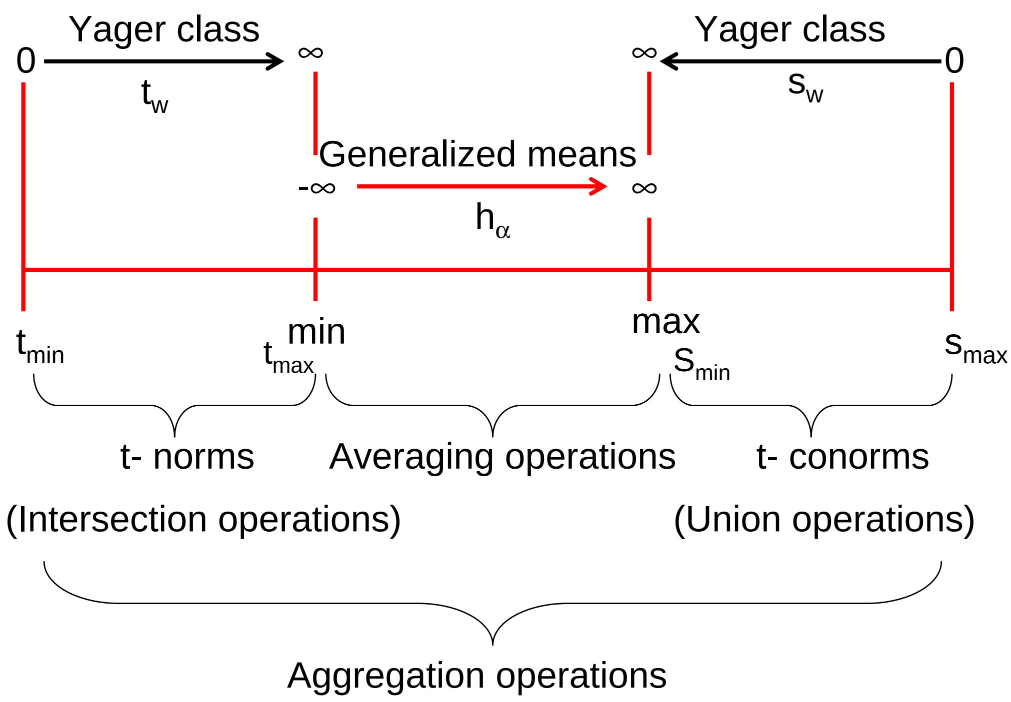
where $a_i = \mu_{A_i}(x)$, $i = 1, 2, \ldots\ldots\ldots, n$

e.g. generalized means:

$$h_\alpha(a_1, a_2, \cdots, a_n) \triangleq \left(\frac{a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha}{n} \right)^{1/\alpha}, \quad \alpha \in R. \ \alpha \ne 0$$

$\alpha \to -\infty$ h_α become $\min(a_1, a_2, \cdots, a_n)$

$\alpha \to \infty$ h_α become $\max(a_1, a_2, \cdots, a_n)$.



~ *weighted generalized means:*

$$h_\alpha(a_1, a_2, \cdots, a_n \;;\; w_1, w_2, \cdots, w_n) \triangleq \left(w_1 a_1^\alpha + w_2 a_2^\alpha + \cdots + w_n a_n^\alpha\right)^{\frac{1}{\alpha}}$$

$$w_i \geq 0, \quad \sum_{i=1}^{n} w_i = 1.$$

- *Other operations:*

1. Fuzzy conjunction: A∧B

$$\mu_{A\wedge B}(x, y) \triangleq t\left(\mu_A(x), \mu_B(y)\right). \quad \text{t: t-norm.}$$

2. Fuzzy disjunction: A∨B

$$\mu_{A\vee B}(x, y) \triangleq s\left(\mu_A(x), \mu_B(y)\right). \quad \text{s: t-conorm.}$$

3. Fuzzy implication: A→B

① Material implication : $A \to B = s(\bar{A}, B)$

② Propositional calculus : $A \to B = s(\bar{A}, t(A, B))$

③ Extended propositional calculus : $A \to B = s(\bar{A} \times \bar{B}, B)$

④ Generalization of modus porens :

$$A \to B = \sup\{k \in [0,1],\ t(A,k) \leq B\}.$$

⑤ Generalization of modus tollens :

$$A \to B = \inf\{k \in [0,1],\ s(B,k) \leq A\}.$$

Ex: t: algebraic product.

s: algebraic sum.

Material implication is adopted.

$$\mu_{A\wedge B}(x,y)=\mu_A(x)\cdot\mu_B(y)$$

$$\mu_{A\vee B}(x,y)=\mu_A(x)+\mu_B(y)-\mu_A(x)\cdot\mu_B(y).$$

$$\mu_{A\rightarrow B}(x,y)=\mathrm{s}(1\text{-}\mu_A(x),\mu_B(y))$$

$$=1\text{-}\mu_A(x)+\mu_B(y)\ \text{-}(1\text{-}\mu_A(x))\cdot\mu_B(y)$$

$$=1\text{-}\mu_A(x)+\mu_A(x)\cdot\mu_B(y).$$

• ***Extension principle:***

~ extends point-to-point mappings to mappings for fuzzy sets.

~ Given a function f : U → V and a fuzzy set A in U,

where A = $\mu_1/x_1+\mu_2/x_2+\ldots+\mu_n/x_n$.

the extension principle states that

$$f(A)=f(\mu_1/x_1+\mu_2/x_2+\ldots+\mu_n/x_n)$$
$$=\mu_1/f(x_1)+\mu_2/f(x_2)+\ldots+\mu_n/f(x_n)$$

If more than one element of U is mapped to the same element y in V by f (i.e. many-to-one mapping), then the maximum among their membership grades is taken. That is

$$\mu_{f(A)}(y)=\sup_{\substack{x_i\in U\\ f(x_i)=y}}\left[\mu_A(x_i)\right].$$

~ Let U be a Cartesian product of universes $U=U_1\times U_2\times\ldots\times U_n$, and $A_1, A_2,\ldots, A_n$ be n fuzzy sets in $U_1, U_2, \ldots, U_n$

Let $y = f(x_1,x_2, \ldots x_n)$. By extension principle, the function $f(x_1,x_2, \ldots, x_n)$ can be extended to act on the n fuzzy subsets of U, $A_1,A_2, \ldots,A_n$, such that

B = f (A1,A2, … ,An),

where B is the fuzzy image (fuzzy set) of $A_1,A_2, \ldots ,A_n$

Through f (•), B is defined by

$$B=\left\{(y\ ,\ \mu_B(y))\mid\ y=f(x_1,x_2,\cdots x_n),(x_1,x_2,\cdots,x_n)\in U\right\}.$$

$$\text{where}\quad \mu_B(y)=\sup_{\substack{(x_1,x_2,\cdots x_n)\in U\\ y=f(x_1,x_2,\cdots x_n)}}\min\left[\mu_{A_1}(x_1),\mu_{A_2}(x_2),\cdots,\mu_{A_n}(x_n)\right]$$

Ex: U = {1, 2, 3, … 9,10}. A= "Large" is given as

A = "Large" = 0.5/6 + 0.7/7 + 0.8/8 + 0.9/9 + 1/10.

If $y=f(x)=x^2$, by extension principal, B= "Large"^2 can be calculated as :

B=0.5/36+0.7/49+0.8/64+0.9/81+1/100.

Ex: U={-2 , -1 , 0 , 1 , 2}. A= 0.5/-1 + 0.8/0 + 1/1 + 0.4/2.

$y=f(x)=x^2$,

x	$\mu_A(x)$	$y=f(x)=x^2$	$\mu_B(x)$
-1	0.5	1	Max{0.5 , 1.0}=1.0
0	0.8	0	0.8
1	1.0	1	Max{0.5 , 1.0}=1.0
2	0.4	4	0.4

B=1/1 + 0.8/0 + 0.4/4

Ex: $f : U_1 \times U_2 \rightarrow V$, U_1 = {- 1, 0, 1}, U_2 = {- 2, 2}, V = {- 2, - 1, 2, 3}.

$f(x_1 , x_2) = x_1^2+x_2$

A_1: defined on U_1 and A_1 = 0.5/-1 + 0.1/0 + 0.9/1.

A_2: defined on U_2 and A_2 = 0.4/-2 + 1.0/2.

x_1	μ_{A1}	x_2	μ_{A2}	$\mu_{A1\times A2}(x_1 , x_2)$	$y=f(x_1 , x_2)= x_1^2 + x_2$
-1	0.5	-2	0.4	min{0.5 , 0.4}=0.4	-1
-1	0.5	2	1.0	min{0.5 , 1.0}=0.5	3
0	0.1	-2	0.4	min{0.1 , 0.4}=0.1	-2
0	0.1	2	1.0	min{0.1 , 1.0}=0.1	2
1	0.9	-2	0.4	min{0.9 , 0.4}=0.4	-1
1	0.9	2	1.0	min{0.9 , 1.0}=0.9	3

μ_B(y= -1)= max{0.4, 0.4} = 0.4, μ_B(y= -2)= 0.1,

μ_B(y= 3) = max{0.5, 0.9} = 0.9, μ_B(y= 2)= 0.1,

B= 0.1/-2 + 0.4/-1 + 0.1/2 + 0.9/3

Appendix: Factual Information about the Impact of Fuzzy Logic

(data from e-News of Berkeley Initiative in Soft Computing (BISC), Nov. 2004)

~ PATENTS

. Number of fuzzy-logic-related patents applied for in Japan: 17,740
. Number of fuzzy-logic-related patents issued in Japan: 4,801
. Number of fuzzy-logic-related patents issued in the US: around 1,700

~ PUBLICATIONS

Count of papers containing the word "fuzzy" in title, as cited in INSPEC and MATH.SCI.NET databases.
Compiled by Camille Wanat, Head, Engineering Library, UC Berkeley, May, 2006

Number of papers in INSPEC and MathSciNet which have "fuzzy" in their titles:

INSPEC - "fuzzy" in the title		MathSciNet - "fuzzy" in the title	
1970-1979:	569	1970-1979:	443
1980-1989:	2,403	1980-1989:	2,465
1990-1999:	23,210	1990-1999:	5,487
2000-2006(5)	21,147	2000-2006(5):	5,504
Total:	47,329	Total:	13,899

~ JOURNALS ("fuzzy" or "soft computing" in title)

1. Fuzzy Sets and Systems
2. IEEE Transactions on Fuzzy Systems
3. Fuzzy Optimization and Decision Making
4. Journal of Intelligent & Fuzzy Systems
5. Fuzzy Economic Review
6. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems
7. Journal of Japan Society for Fuzzy Theory and Systems
8. International Journal of Fuzzy Systems
9. Soft Computing
10. International Journal of Approximate Reasoning--Soft Computing in Recognition and Search
11. Intelligent Automation and Soft Computing
12. Journal of Multiple-Valued Logic and Soft Computing
13. Mathware and Soft Computing
14. Biomedical Soft Computing and Human Sciences
15. Applied Soft Computing

~ Applications

The range of application-areas of fuzzy logic is too wide for exhaustive listing. Following is a partial list of existing application-areas in which there is a record of substantial activity.

1. Industrial control
2. Quality control
3. Elevator control and scheduling
4. Train control
5. Traffic control
6. Loading crane control
7. Reactor control
8. Automobile transmissions
9. Automobile climate control
10. Automobile body painting control
11. Automobile engine control
12. Paper manufacturing
13. Steel manufacturing
14. Power distribution control
15. Software engineering
16. Expert systems
17. Operation research
18. Decision analysis
19. Financial engineering
20. Assessment of credit- worthiness
21. Fraud detection
22. Mine detection
23. Pattern classification
24. Oil exploration
25. Geology
26. Civil Engineering
27. Chemistry
28. Mathematics
29. Medicine
30. Biomedical instrumentation
31. Health-care products
32. Economics
33. Social Sciences
34. Internet
35. Library and Information Science

~ Product Information

1. Information from SIEMENS:

- washing machines, 2 million units sold
- fuzzy guidance for navigation systems (Opel, Porsche)
- OCS: Occupant Classification System (to determine, if a place in a car is occupied by a person or something else; to control the airbag as well as the intensity of the airbag). Here FL is used in the product as well as in the design process (optimization of parameters).
- fuzzy automobile transmission (Porsche, Peugeot, Hyundai)

2. Information from OMRON:

fuzzy logic blood pressure meter, 7.4 million units sold, approximate retail value $740 million dollars

3. Facts on FL-based systems in Japan (as of 2/06/2004)

- Sony's FL camcorders

Total amount of camcorder production of all companies in 1995-1998 times. Sony's market share is the following. Fuzzy logic is used in all Sony's camcorders at least in these four years, i.e. total production of Sony's FL-based camcorders is 2.4 millions products in these four years.

1,228K units X 49% in 1995
1,315K units X 52% in 1996
1,381K units X 50% in 1997
1,416K units X 51% in 1998

• FL control at Idemitsu oil factories

Fuzzy logic control is running at more than 10 places at 4 oil factories of Idemitsu Kosan Co. Ltd including not only pure FL control but also the combination of FL and conventional control.

They estimate that the effect of their FL control is more than 200 million YEN per year and it saves more than 4,000 hours per year.

• Canon

Canon used (uses) FL in their cameras, camcorders, copy machine, and stepper alignment equipment for semiconductor production. But, they have a rule not to announce their production and sales data to public.

Canon holds 31 and 31 established FL patents in Japan and US, respectively.

• Minolta cameras

Minolta has a rule not to announce their production and sales data to public, too, whose name in US market was Maxxum 7xi. It used six FL systems in a camera and was put on the market in 1991 with 98,000 YEN (body price without lenses). It was produced 30,000 per month in 1991. Its sister cameras, alpha-9xi, alpha-5xi, and their successors used FL systems, too. But, total number of production is confidential.

• FL plant controllers of Yamatake Corporation

Yamatake-Honeywell (Yamatake's former name) put FUZZICS, fuzzy software package for plant operation, on the market in 1992. It has been used at the plants of oil, oil chemical, chemical, pulp, and other industries where it is hard for conventional PID controllers to describe the plan process for these more than 10 years.

They planed to sell the FUZZICS 20 - 30 per year and total 200 million YEN.

As this software runs on Yamatake's own control systems, the software package itself is not expensive comparative to the hardware control systems.

• Others

Names of 225 FL systems and products picked up from news articles in 1987 - 1996 are listed at http://www.adwin.com/elec/fuzzy/note_10.html in Japanese.)